Characterization and Automated Computation of Invariant Algebraic Sets for Algebraic Differential Equations

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NYU, New York, USA
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Context: Hybrid Systems Model

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Sensing:  read data from sensors
```

Context: Hybrid Systems Model

```
Sensing:  read data from sensors
Control:  actuate
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Plant:  evolve
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Context: Hybrid Systems Model

Init

$\longrightarrow$

(

Sensing: read data from sensors

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Safety

Context: Hybrid Systems Model

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\begin{array}{l}
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\longrightarrow \\
[\\
(\\
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)^* \\
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Context: Hybrid Systems Model

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Plant: evolve ◀ ◀ ◀ ◀ ◀

)*

]

Safety

Evolution

- Continuous time
- Ordinary Differential Equations (ODE)

Related work: Handling the continuous part

- Solutions are hard to compute symbolically
- No closed form solution exists in general
- Alternatives
  - Local approximations (Taylor series) [Lanotte et al. 2005]
  - Inductive (differential) Invariants [Maths, ThPhy 1870-] [Control 1900-] [FM 2001-]
- Limitations
  - Linear differential equations [Tiwari et al. 2003-]
  - Restrictive subclasses of Invariants [Sankaranarayanan et al 2006-, Matringe et al. 2009-, Platzer 2010]
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This talk

Polynomial

All Algebraic Sets

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This talk

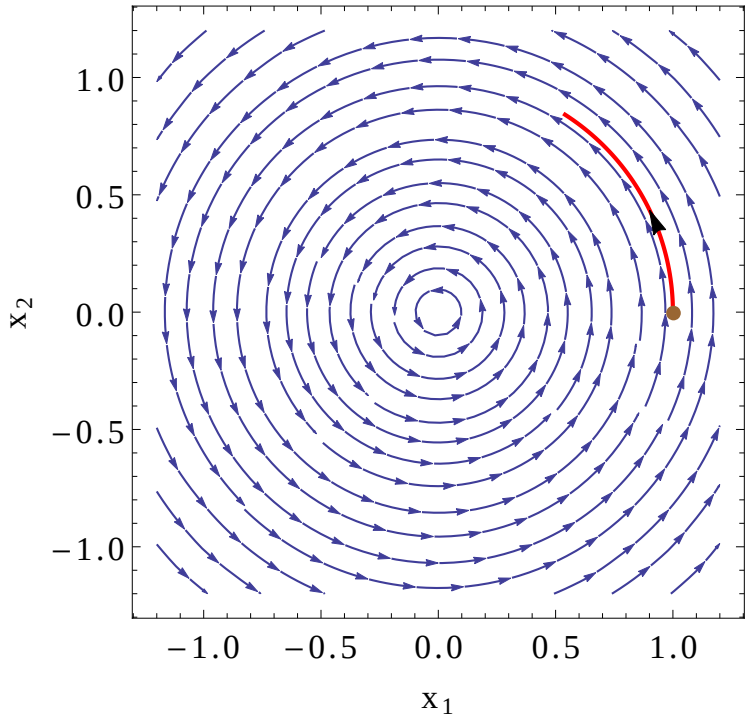
Polynomial

All Algebraic Sets

Efficient

Algebraic Invariant Equations

$$(\dot{x_1}, \dot{x_2}) = (-x_2, x_1)$$



The solution for $\mathbf{x}_0 = (1, 0)$ for $t = [0, 1]$

Algebraic Invariant Equations

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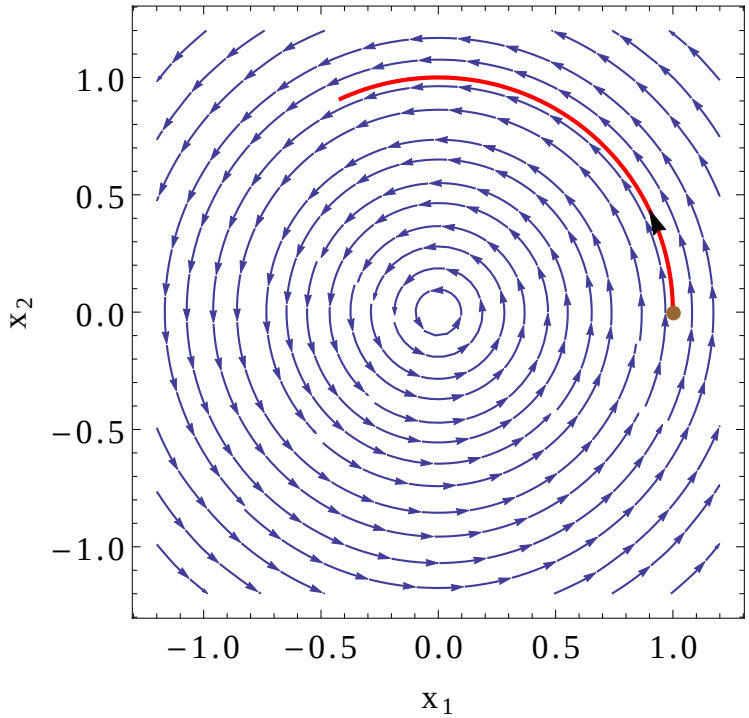

The solution for $\mathbf{x}_0 = (1, 0)$ for $t = [0, 2]$

Algebraic Invariant Equations

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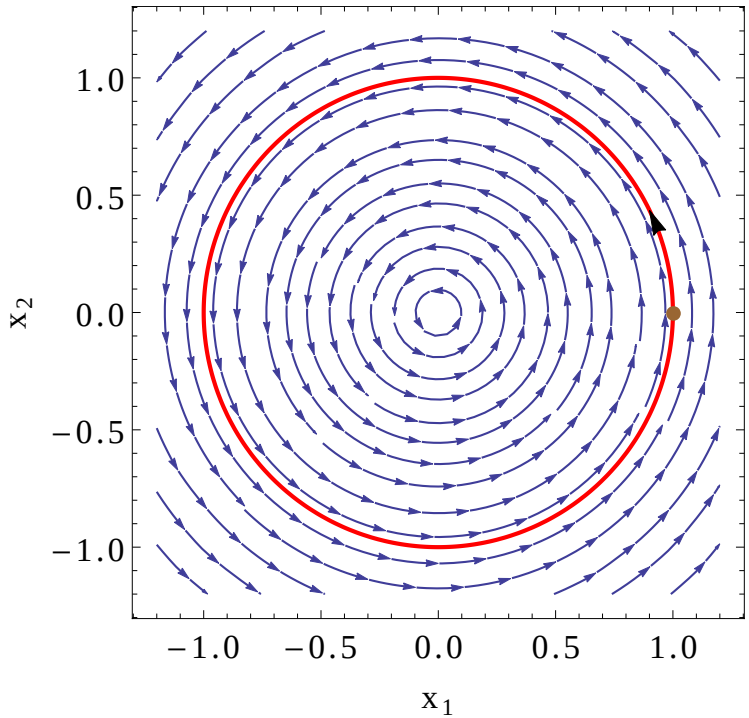
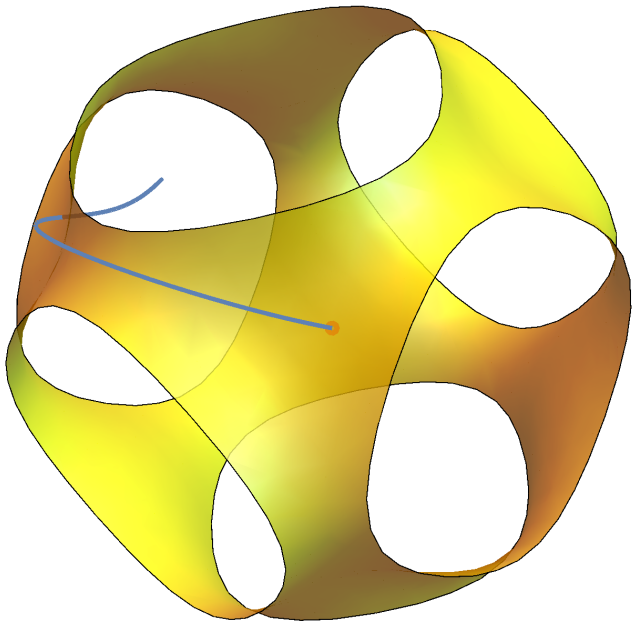


Algebraic Invariant Equation

The solution for $\mathbf{x}_0 = (1, 0)$ respects $\boxed{x_1(t)^2 + x_2(t)^2 - 1 = 0}$ $\forall t$

Algebraic Sets, 3D example



Ordinary Differential Equation

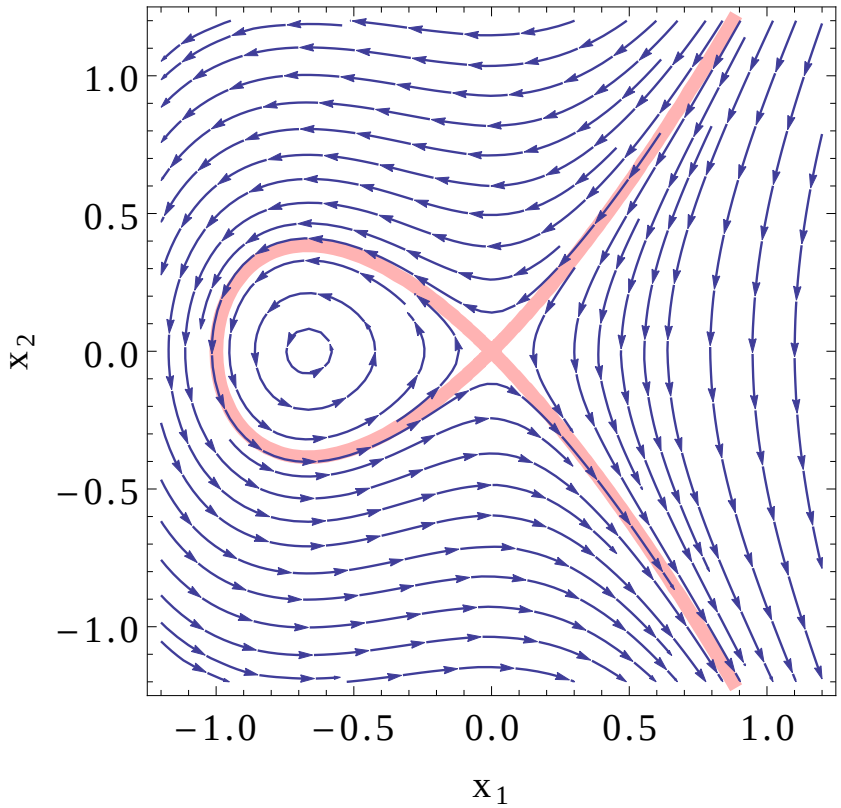
$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} yz \\ -xz \\ -xy \end{pmatrix} = \mathbf{f}$$

Algebraic Sets

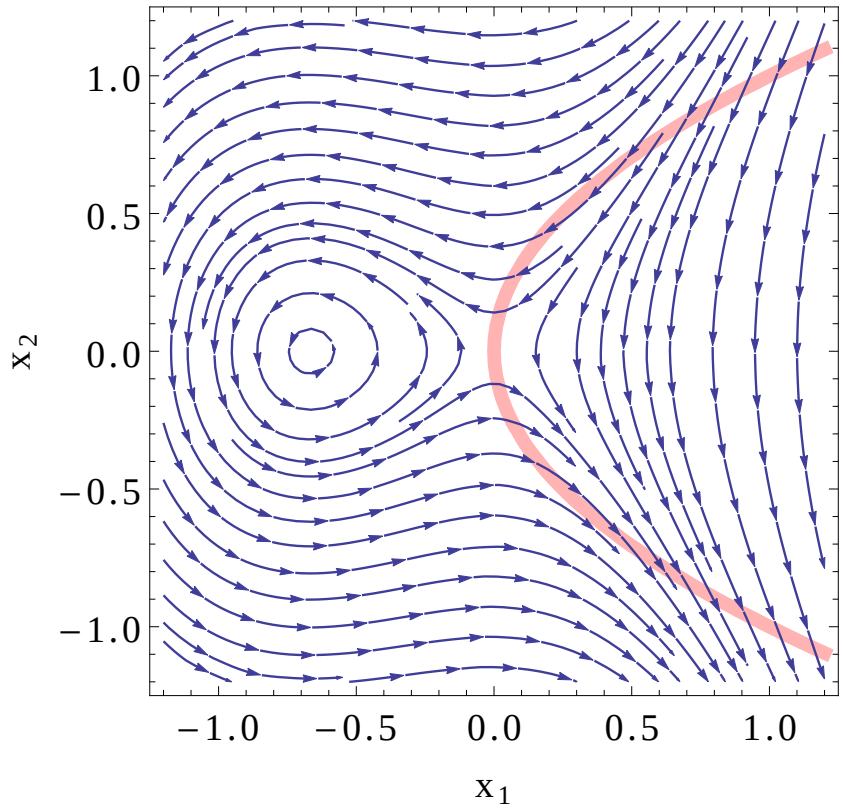
$$S = \{(x, y, z) \mid \underbrace{3x^2 + 3y^2 - 2x^2y^2 + 3z^2 - 2x^2z^2 - 2y^2z^2}_{p(x,y,z)} = 0\}$$

Problem I. *Checking the invariance of Algebraic Equations*

Given $\dot{\mathbf{x}} = \mathbf{p}$, and $\mathbf{x}_0$ such that $h(\mathbf{x}_0) = 0$, is $h(\mathbf{x}(t)) = 0$ for all t ?



$$h(x_1, x_2) = x_1^2 + x_1^3 - x_2^2$$



$$h(x_1, x_2) = x_1 - x_2^2$$

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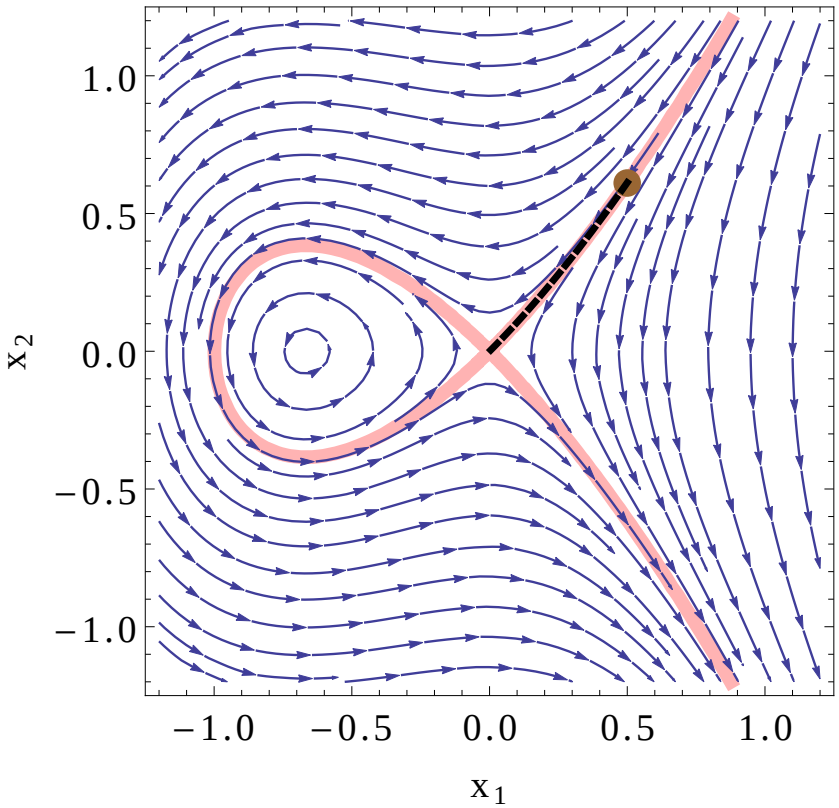

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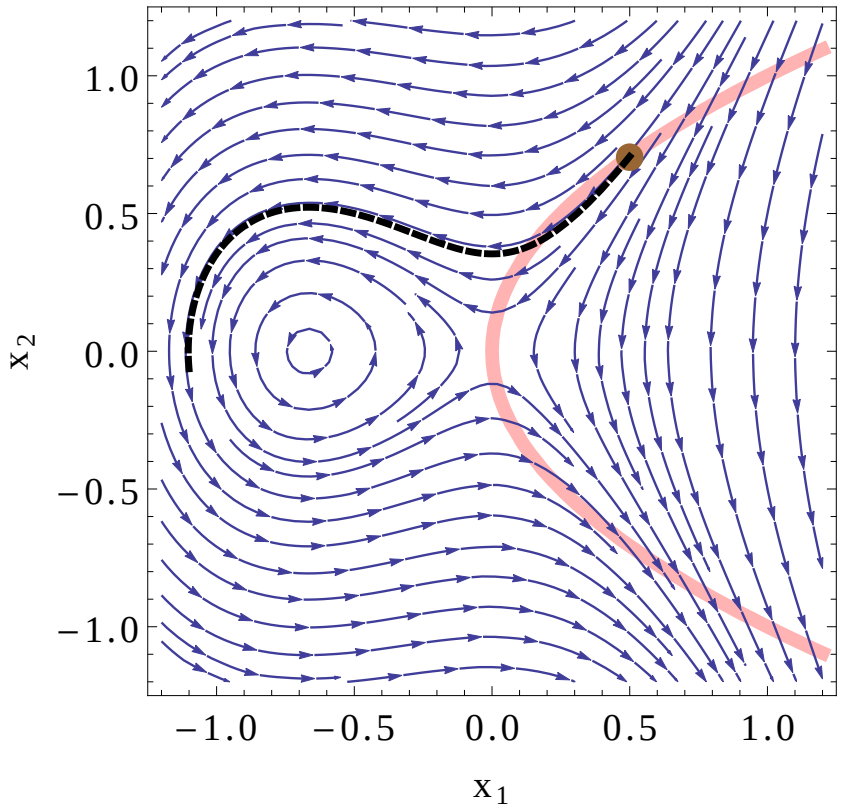

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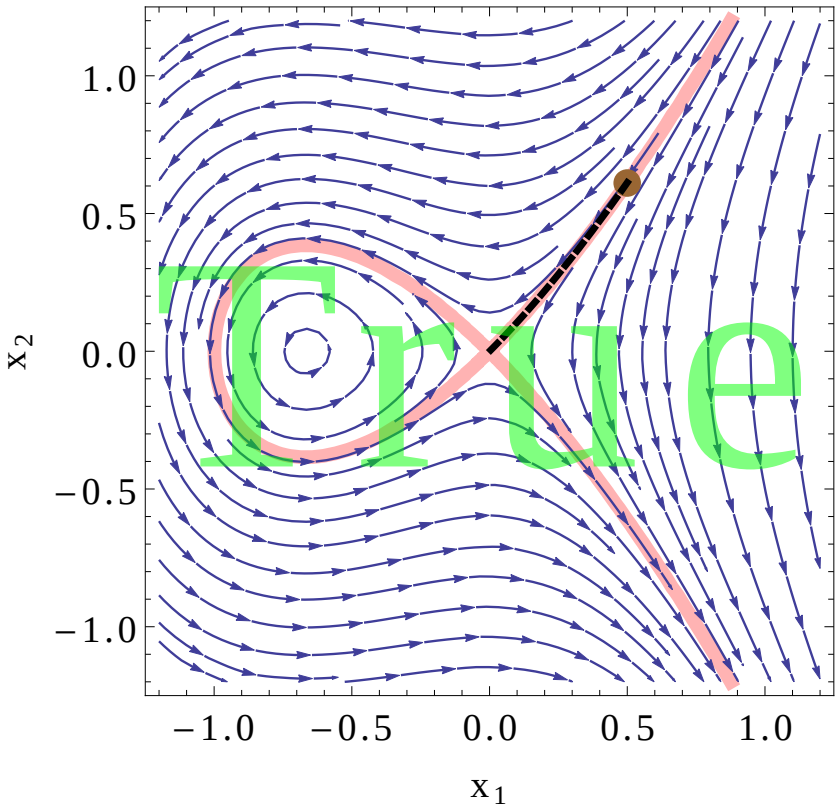

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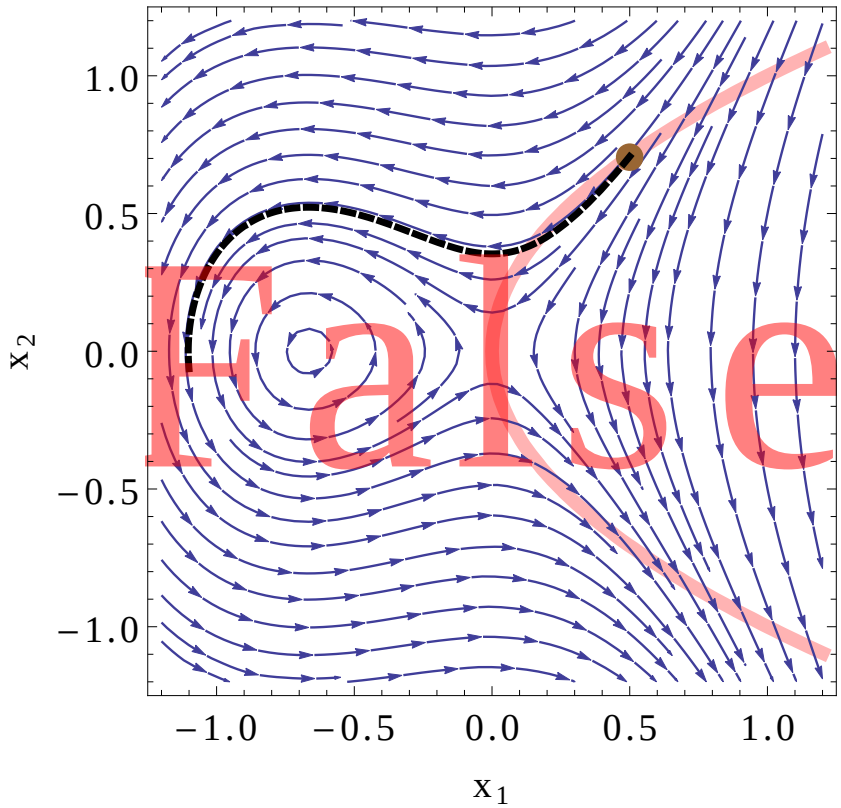


$$h(x_1, x_2) = x_1 - x_2^2$$

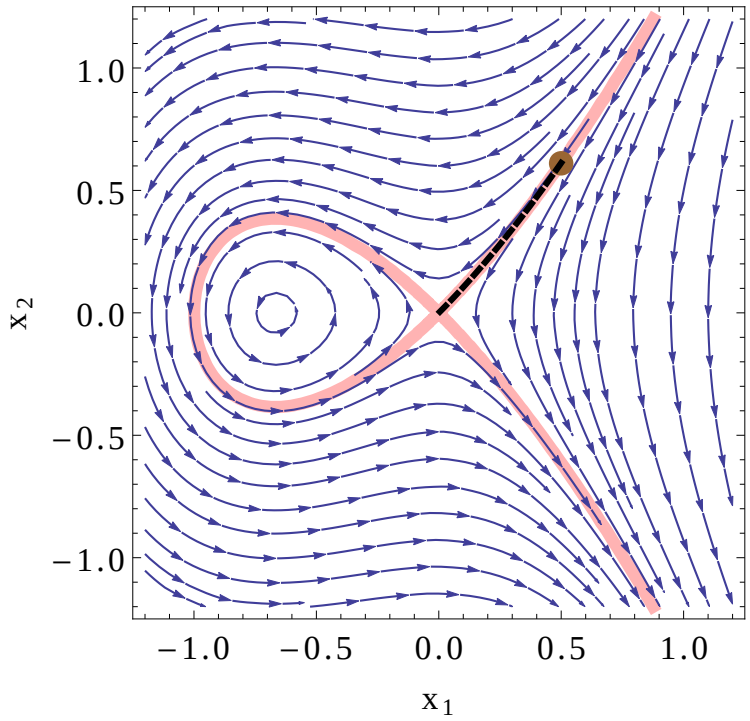
Abstracting Orbits Using Algebraic Sets

Concrete Domain

The trajectory of the solution of an Initial Value Problem ($\dot{\mathbf{x}} = \mathbf{p}$, $\mathbf{x}_0$).

Abstract Domain

Algebraic Sets.

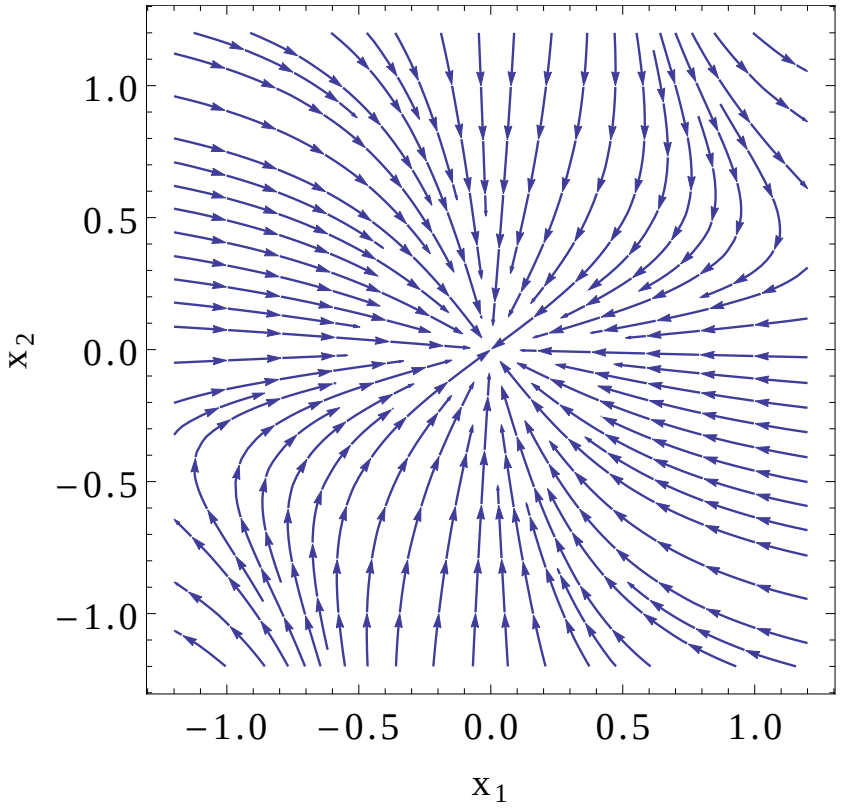


Problem: Checking soundness

Checking the soundness of the abstraction: does a given algebraic set overapproximate the trajectory of the solution ?

Problem II. *Generate Algebraic Invariant Equations*

Given $\dot{\mathbf{x}} = \mathbf{p}$, how to generate h such that $h(\mathbf{x}(t)) = 0$?



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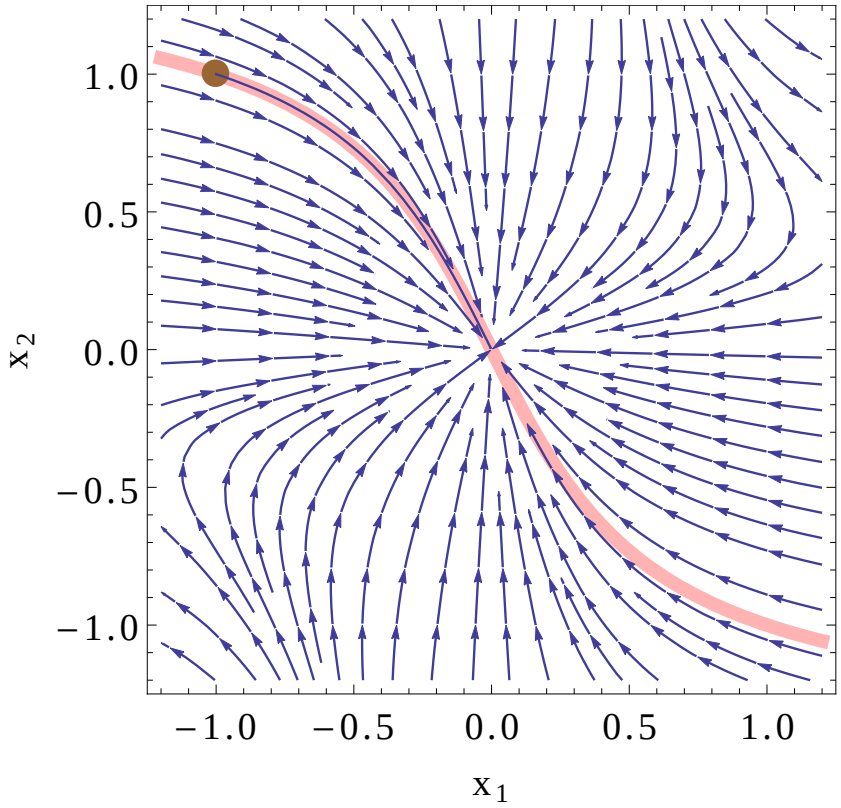

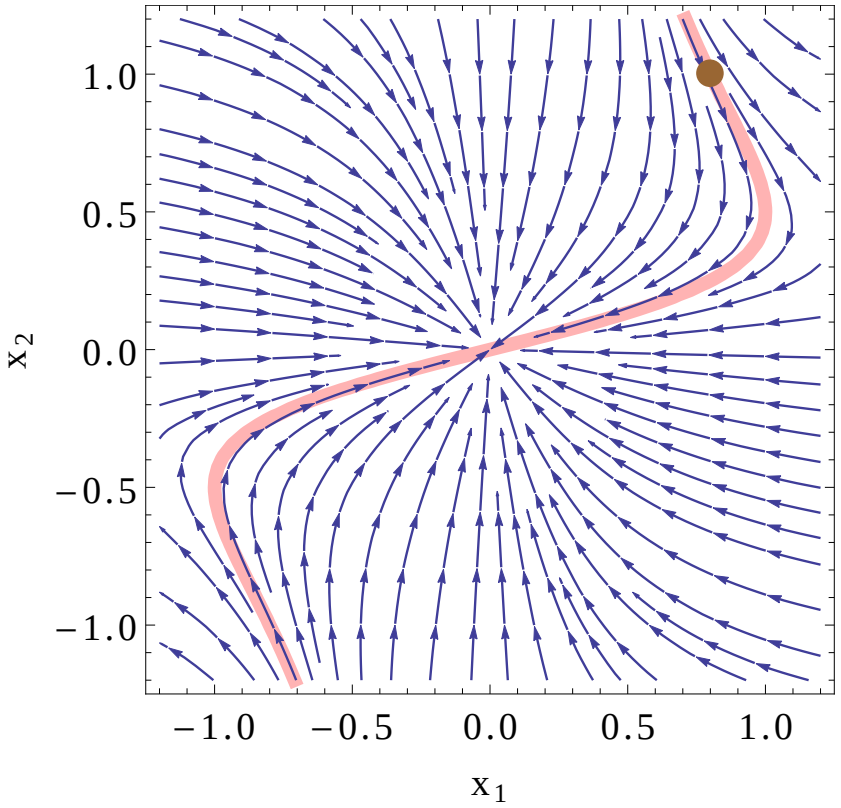

$$h_{(x_1(0),x_2(0))}(x_1, x_2) = (x_2(0) - x_1(0)x_2(0)^2)x_1 - x_1(0)(x_2 - x_1x_2^2)$$

Outline

Some definitions

Lie Derivative along a vector field $\dot{\mathbf{x}} = \mathbf{p}$

$$\mathfrak{D}(h) \stackrel{\text{def}}{=} \sum_{i=1}^{n} \frac{\partial h}{\partial x_i}\dot{\mathbf{x}}_i = \sum_{i=1}^{n} \frac{\partial h}{\partial x_i}\mathbf{p}_i = \frac{dh(\mathbf{x}(t))}{dt}$$

Higher-order Lie derivatives:

$$\mathfrak{D}^{(k+1)}(h) = \mathfrak{D}(\mathfrak{D}^{(k)}(h))$$

Ideal Membership

$$\exists \lambda_i \in \mathbb{R}[\mathbf{x}] : h = \lambda_1 q_1 + \cdots + \lambda_r q_r \quad \leftrightarrow \quad h \in \langle q_1, \ldots, q_r \rangle$$

Ideal membership can be checked effectively using Gröbner bases algorithm.

Checking Invariance of Candidates

Already existing proof rules

I. Checking the invariance of Algebraic Equations

Given $\dot{\mathbf{x}} = \mathbf{p}$, and $\mathbf{x}_0$ such that $h(\mathbf{x}_0) = 0$, is $h(\mathbf{x}(t)) = 0$ for all t ?

$$\mathfrak{D}(h) = \lambda h \quad (\lambda \in \mathbb{R}[\mathbf{x}])$$

$$\frac{\mathfrak{D}(h) = 0}{(h = 0) \rightarrow [\dot{\mathbf{x}} = \mathbf{p}](h = 0)}$$

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- $\mathfrak{D}(h) = 2x_1\mathfrak{D}(x_1) + 2x_2\mathfrak{D}(x_2) = 4x_1x_2$ (Chain Rule)
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$$\mathfrak{D}^{(2)}(h) = 4h$$

No $\lambda \in \mathbb{R}[x]$ s.t. $4x_1x_2 = \lambda(x_1^2 + x_2^2)$

$$\mathfrak{D}(h) = 4x_1x_2 \neq 0$$

$$(h = 0) \rightarrow [\dot{\mathbf{x}} = \mathbf{p}](h = 0)$$

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still unsound! (counterexample: $h = x_1$)

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Checking Invariance of Candidates

Differential Radical Invariants [Theorem 2, TACAS'14]

$$\mathfrak{D}^{(N_p)}(p) \in \langle p, \ldots, \mathfrak{D}^{(N_p-1)}(p)\rangle \ \wedge \ p = 0 \rightarrow \mathfrak{D}^{(N_p-1)}(p) = 0$$

$$\vdots$$

$$\mathfrak{D}^{(3)}(p) \in \langle p, \mathfrak{D}(p), \mathfrak{D}^{(2)}(p)\rangle \ \wedge \ p = 0 \rightarrow \mathfrak{D}^{(2)}(p) = 0$$

$$\mathfrak{D}^{(2)}(p) \in \langle p, \mathfrak{D}(p)\rangle \ \wedge \ p = 0 \rightarrow \mathfrak{D}(p) = 0$$

$$\frac{\mathfrak{D}(p) \in \langle p\rangle \ (\exists \lambda \in \mathbb{R}[\mathbf{x}] : \mathfrak{D}(p) = \lambda p)}{(p = 0) \rightarrow [\dot{\mathbf{x}} = \mathbf{f}](p = 0)}$$

- order N_p is **finite**: **unknown** a priori and **computed on the fly**
- $< N_p$ ideal membership problems: $\mathfrak{D}^{(i+1)}(p) \in \langle p, \ldots, \mathfrak{D}^{(i)}(p)\rangle$
- $< N_p - 1$ quantifier elimination problems: $p = 0 \rightarrow \mathfrak{D}^{(i)}(p) = 0$

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$$\text{✗}\mathfrak{D}(p) \in \langle p\rangle \ (\exists \lambda \in \mathbb{R}[\mathbf{x}] : \mathfrak{D}(p) = \lambda p)$$

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Differential Radical Invariants [Theorem 2, TACAS'14]

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$$\vdots$$

$$\mathfrak{D}^{(3)}(p) \in \langle p, \mathfrak{D}(p), \mathfrak{D}^{(2)}(p)\rangle \ \wedge \ p = 0 \rightarrow \mathfrak{D}^{(2)}(p) = 0$$

$$\text{✗}\mathfrak{D}^{(2)}(p) \in \langle p, \mathfrak{D}(p)\rangle \ \wedge \ p = 0 \rightarrow \mathfrak{D}(p) = 0\text{✓}$$

$$\text{✗}\mathfrak{D}(p) \in \langle p\rangle \ (\exists \lambda \in \mathbb{R}[\mathbf{x}] : \mathfrak{D}(p) = \lambda p)$$

$$\frac{}{(p = 0) \rightarrow [\dot{\mathbf{x}} = \mathbf{f}](p = 0)}$$

- order N_p is **finite**: **unknown** a priori and **computed on the fly**
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Naïve Approach: DRI + Sum of Squares

$$p = 0 \wedge q = 0$$

$$\equiv_{\mathbb{R}}$$

$$p^2 + q^2 = 0$$

$$(\text{SoSDRI})\frac{(p^2 + q^2 = 0) \rightarrow [\dot{\mathbf{x}} = \mathbf{f}](p^2 + q^2 = 0)}{(p = 0 \wedge q = 0) \rightarrow [\dot{\mathbf{x}} = \mathbf{f}](p = 0 \wedge q = 0)}$$

Drawback: Bad complexity

- \+ Already decides algebraic invariants
- – Increases total polynomial degree

Conjunctive Differential Radical Characterization

Theorem 2, Algorithm 1, SAS'14

$$\mathfrak{D}^{(N_{p,q})}(p), \mathfrak{D}^{(N_{p,q})}(q) \in \langle p, q, \ldots, \mathfrak{D}^{(N_{p,q}-1)}(p), \mathfrak{D}^{(N_{p,q}-1)}(q) \rangle$$
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- $N_{p,q}$ is a **unique** order for the **entire** conjunction, **shared** between p and q
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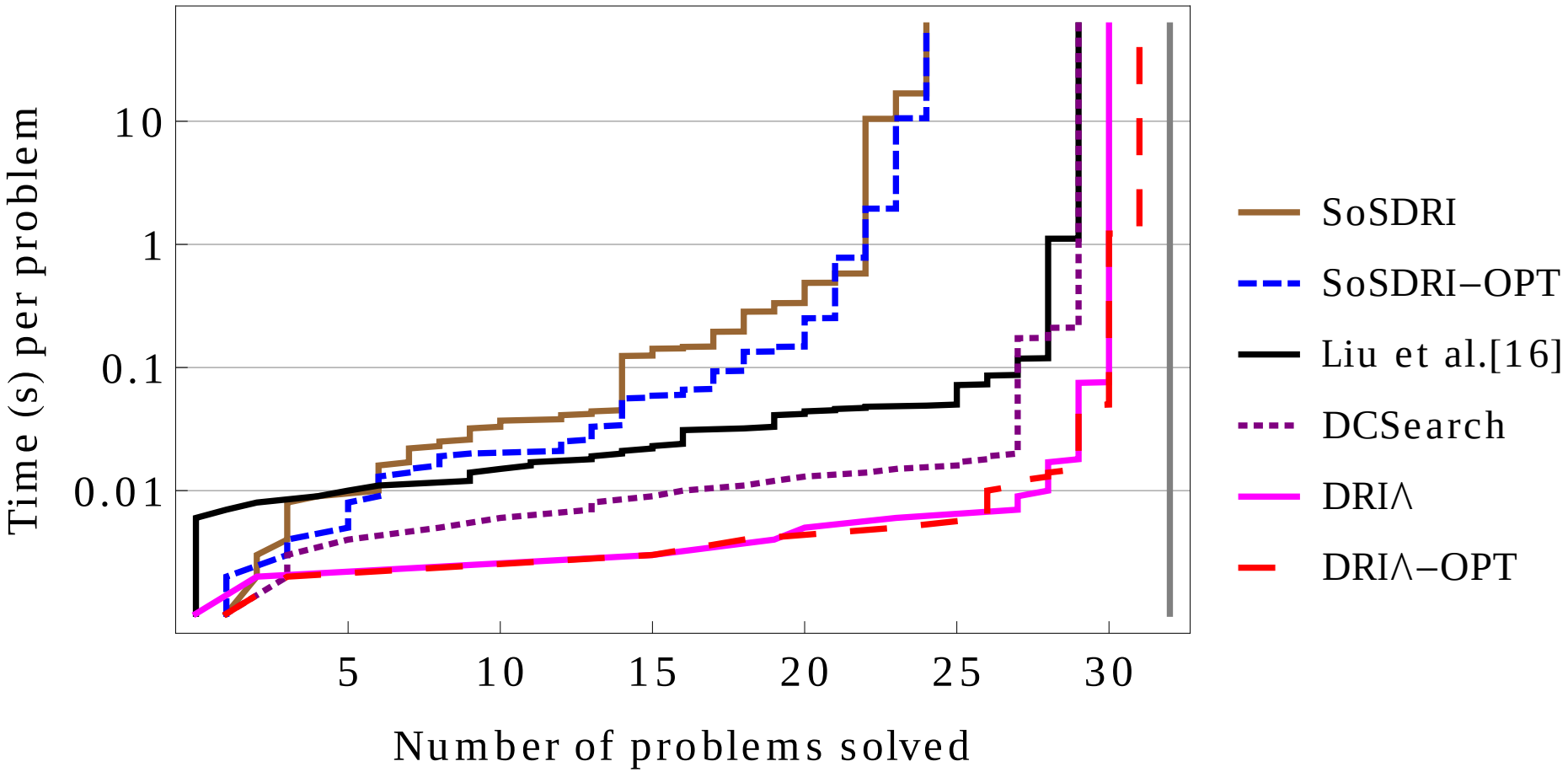
Empirical Performance, Complete Comparison



Outline

Generation of Invariant Algebraic Sets

Necessary and sufficient condition [Theorem 3, TACAS'14]

II. Generate Algebraic Invariant Equations

Given $\dot{\mathbf{x}} = \mathbf{p}$, how to generate h such that $h(\mathbf{x}(t)) = 0$?

Theorem

$S \in \mathbb{R}^n$ is an invariant algebraic set **if and only if**

$$S = \text{ Set of roots of the system } \begin{cases} h & = 0 \\ & \vdots \\ \mathfrak{D}^{(N-1)}(h) & = 0 \end{cases}$$

for some polynomial h with **order** N, that is

$$\mathfrak{D}^{(N)}(h) = \sum_{i=0}^{N-1} \lambda_i \mathfrak{D}^{(i)}(h)$$

Generation of Invariant Algebraic Sets

First integrals vs. Local invariant regions [Theorem 4, TACAS'14]

Suppose we found h and N such that

$$\mathfrak{D}^{(N)}(h) = \sum_{i=0}^{N-1} \lambda_i \mathfrak{D}^{(i)}(h)$$

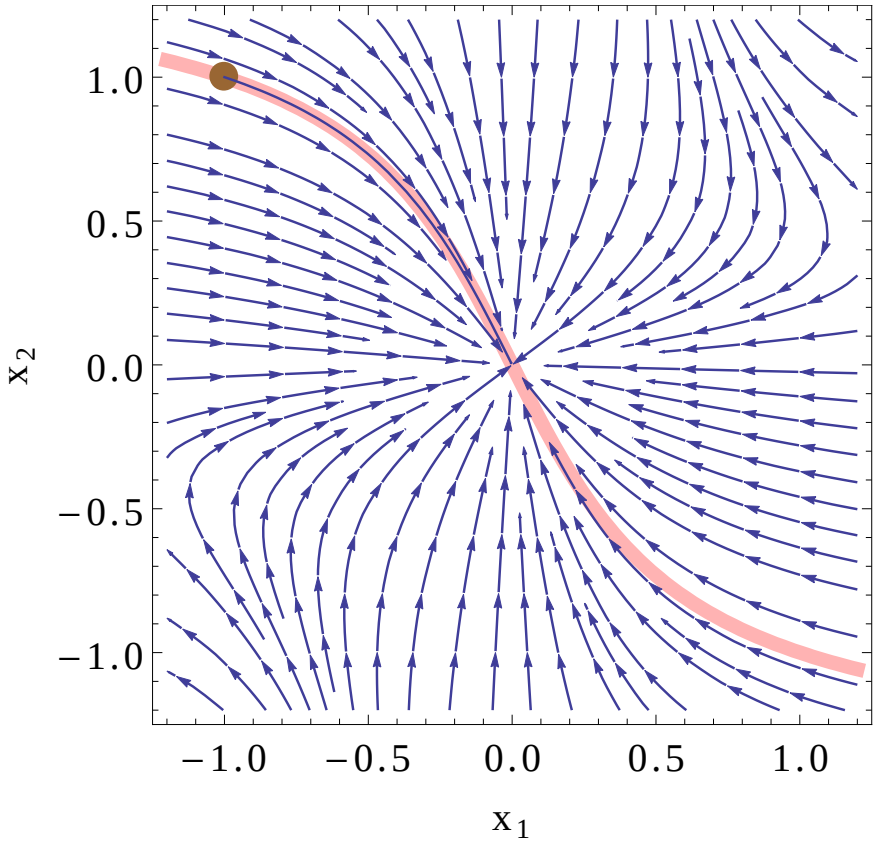
Case 1: First Integral

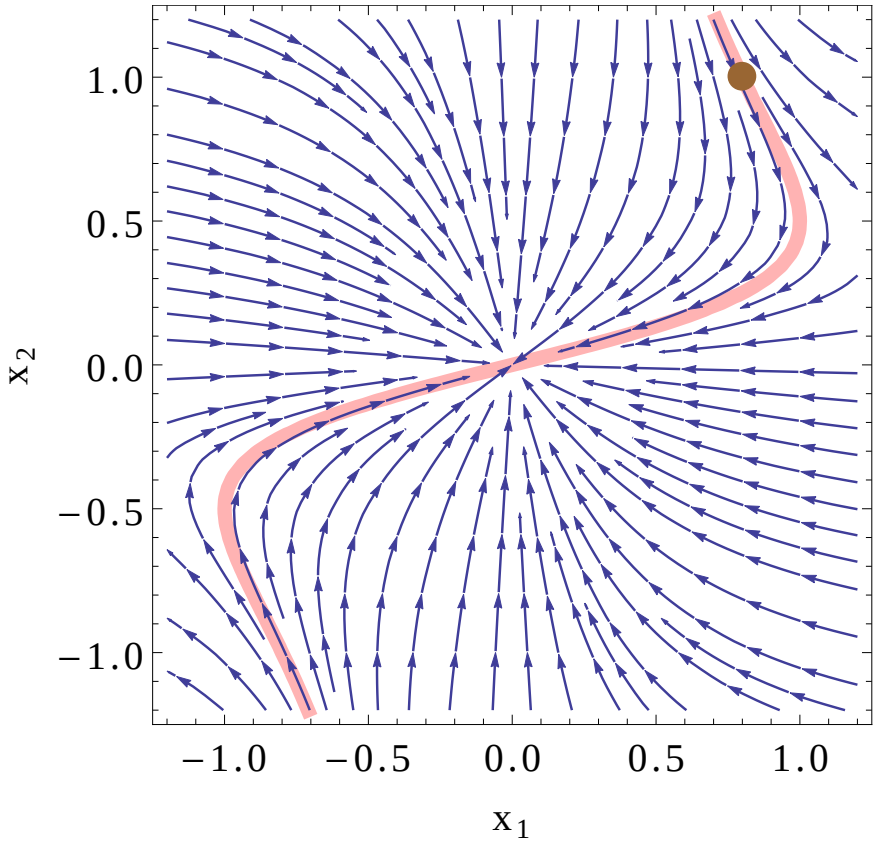
For all $\mathbf{x}_0 \in \mathbb{R}^n$, $h(\mathbf{x}_0) = 0 \wedge \cdots \wedge \mathfrak{D}^{(N-1)}(h)(\mathbf{x}_0) = 0$

Case 2: Local Invariant Regions (e.g. limiting cycle, equilibria)

Restrict $\mathbf{x}_0$ such that $h(\mathbf{x}_0) = 0 \wedge \cdots \wedge \mathfrak{D}^{(N-1)}(h)(\mathbf{x}_0) = 0$

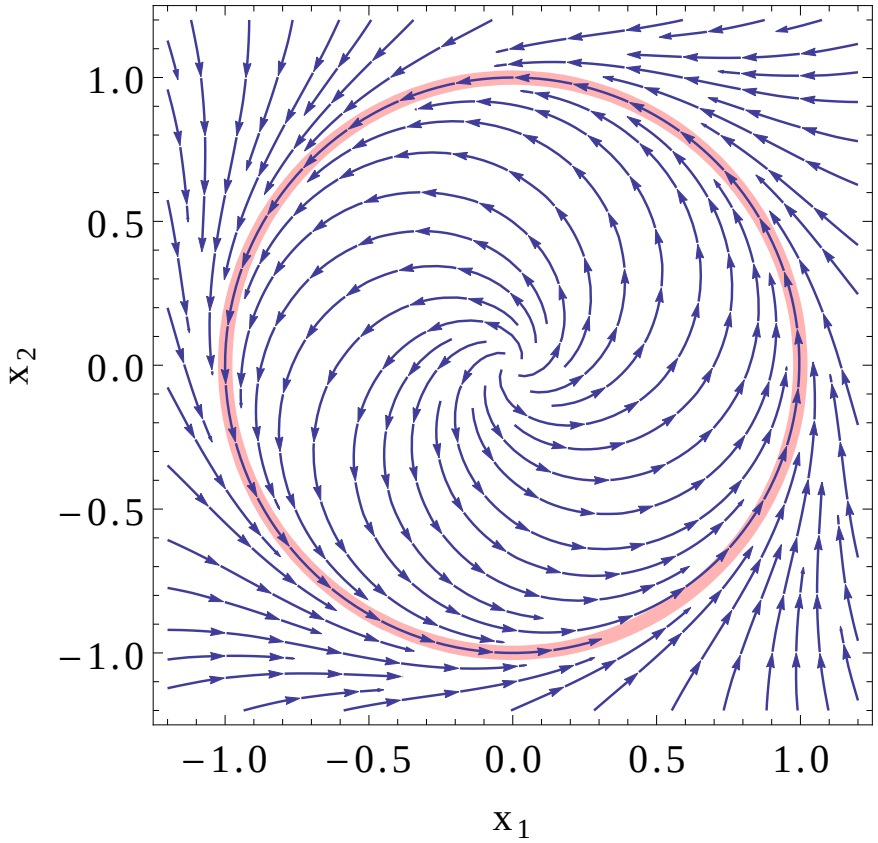
Example: First Integrals



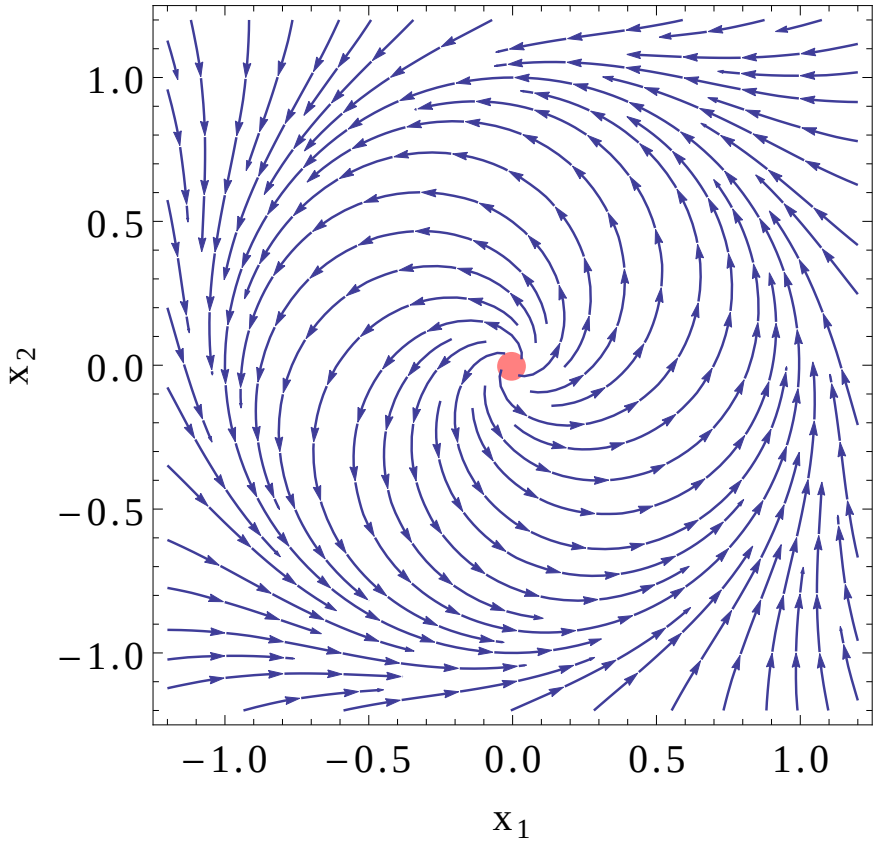


$$h_{(x_1(0),x_2(0))}(x_1,x_2) = (x_2(0)-x_1(0)x_2(0)^2)x_1 - x_1(0)(x_2 - x_1x_2^2)$$

Example: Local invariant regions



$$h(x_1, x_2) = x_1^2 + x_2^2 - 1$$



$$h(x_1, x_2) = x_1^2 + x_2^2$$

But ...

How to **generate** h and N such that

$$\mathfrak{D}^{(N)}(h) = \sum_{i=0}^{N-1} \lambda_i \mathfrak{D}^{(i)}(h)$$

Matrix Representation: Intuition

Suppose we have a 2-dimensional ODE $(\dot{x}_1, \dot{x}_2) = (x_1, x_2)$

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1. Start with parametric h of degree 1: $h = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3$

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$$\begin{array}{ll} (-1+\beta)\alpha_1 & = 0 \\ (-1+\beta)\alpha_2 & = 0 \\ (\beta)\alpha_3 & = 0 \end{array} \leftrightarrow \begin{pmatrix} -1+\beta & 0 & 0 \\ 0 & -1+\beta & 0 \\ 0 & 0 & \beta \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = 0$$

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Study the **null space** (kernel) of $M(\beta)$

Symbolic Linear Algebra

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$$h = x_2(0)x_1 - x_1(0)x_2$$

Toward a Generation Procedure ?

We started with a parametrized polynomial h of degree 1 and $N = 1$ ...

If no invariants:

- Increase order N versus increase the polynomial degree of h ?
- Any bound on N ?
- Any bound on the degree of h ?

Outline

Case Study: Collision Avoidance System

[JAIS'14]

Hybrid System

$$\texttt{pilot} \equiv \omega_1 := *;$$
$$\texttt{cas} \equiv (\texttt{pilot}; ?\texttt{monitor}; \texttt{evolve})^*$$

Safety Property

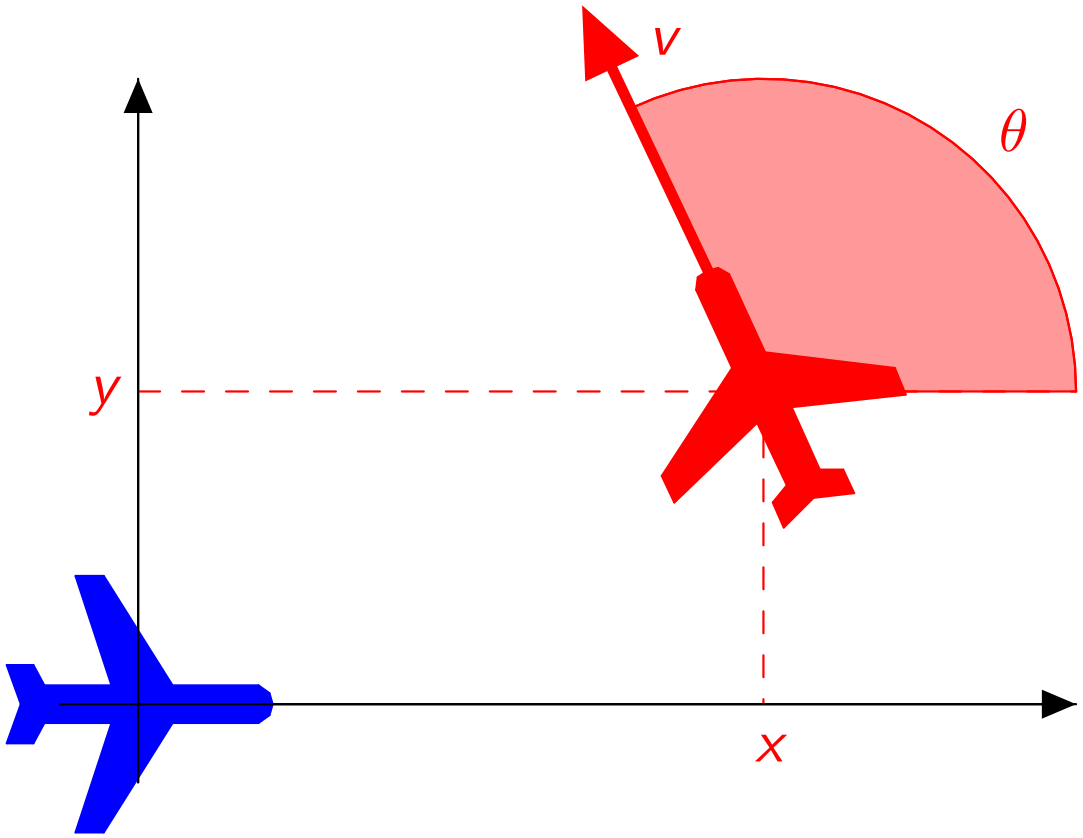
$$\texttt{safety} \equiv \forall\, t \geq 0,\ r(t) > p$$

Challenge: Synthesize `monitor` s.t.

The following d$\mathcal{L}$ formula, expressing that the safety property always holds for the whole system, is valid, i.e., true in all states:

$$\texttt{safety} \rightarrow [\texttt{cas}]\,\texttt{safety},$$

Evolution equations



$$\dot{x} = v_2 \cos\theta - v_1 + \omega_1 y$$
$$\dot{y} = v_2 \sin\theta - \omega_1 x$$
$$\dot{\theta} = \omega_2 - \omega_1$$

Synthesizing the monitor

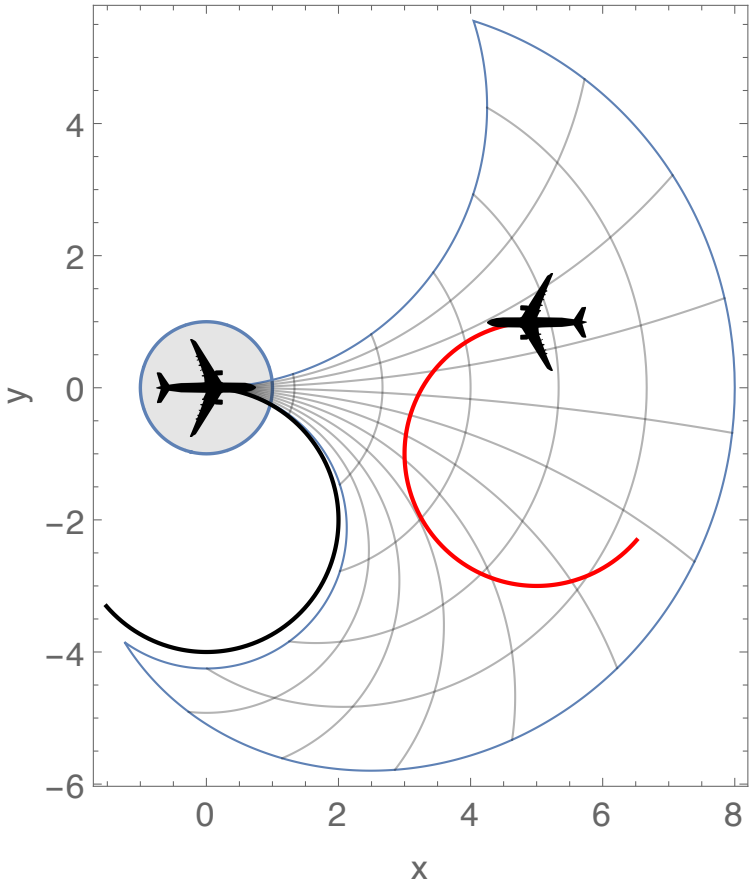
$$\begin{aligned}\texttt{monitor} \equiv &(\omega_1,\omega_2)=(0,0) \longrightarrow I_1(0) > p(v_1+v_2) \wedge \\ &(\omega_1,\omega_2)\neq(0,0) \longrightarrow I_2(0) > 2v_1v_2 \\ &\qquad + 2p(v_2|\omega_1| + v_1|\omega_2|) + p^2|\omega_1\omega_2|\end{aligned}$$

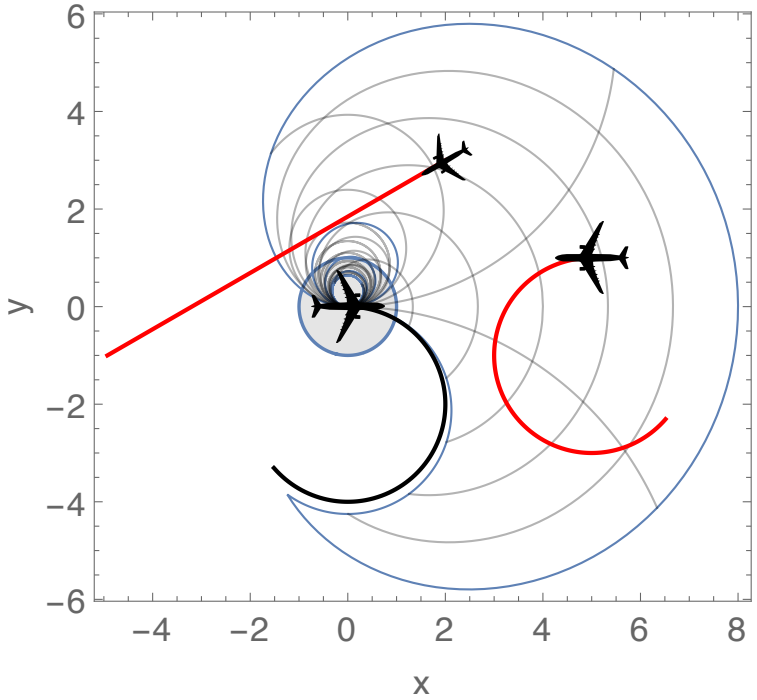
Automatically generated invariants

$$I_1(t) \overset{\text{def}}{=} (v_2\sin\theta(0))x(t) - (v_2\cos\theta(0) - v_1)y(t) = I_1(0)$$

$$\begin{aligned}I_2(t) \overset{\text{def}}{=} &-\omega_1\omega_2(x(t)^2+y(t)^2) + 2v_2\omega_1\sin\theta(t)x(t) \\ &+ 2(v_1\omega_2 - v_2\omega_1\cos\theta(t))y(t) + 2v_1v_2\cos\theta(t) = I_2(0)\end{aligned}$$

Proven safe choices for the ownship





Conclusion

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- **Invariance** of Algebraic Sets is **Decidable**
- **DRI** Necessary and Sufficient **Proof Rule**

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- **Generation** Problem $\sim$ Symbolic **Linear Algebra**
- Equivalent to the Min Rank Problem: **NP-hard**

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- Real Algebraic Geometry $\leftrightharpoons$ Logic $\leftrightharpoons$ Verification

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Algebraic Sets Embedding

$$\underset{\text{Reachable Set}}{\mathcal{O}(\mathbf{x}_0)} \xrightarrow{I} \underset{\text{Vanishing Ideal}}{I(\mathcal{O}(\mathbf{x}_0))} \xrightarrow{V} \underset{\text{Closure}}{V(I(\mathcal{O}(\mathbf{x}_0)))} \supseteq \underset{\text{Reachable Set}}{\mathcal{O}(\mathbf{x}_0)}$$

Vanishing Ideal: all polynomials that vanish on $\mathcal{O}(\mathbf{x}_0)$

$$I(\mathcal{O}(\mathbf{x}_0)) \stackrel{\text{def}}{=} \{h \in \mathbb{R}[\mathbf{x}] \mid \forall \mathbf{x} \in \mathcal{O}(\mathbf{x}_0), h(\mathbf{x}) = 0\}$$

Closure: common roots of all polynomials in I

$$V(I) \stackrel{\text{def}}{=} \{\mathbf{x} \in \mathbb{R}^n \mid \forall h \in I, h(\mathbf{x}) = 0\}$$

Soundness: $V(I(\mathcal{O}(\mathbf{x}_0)))$ is the **smallest variety** that **contains** $\mathcal{O}(\mathbf{x}_0)$

Algebraic Sets Embedding

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Limitation: Zariski Dense Varieties

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$\dot{x} = x \rightsquigarrow \mathcal{O}(\mathbf{x}_0) = [\mathbf{x}_0, \infty[\rightsquigarrow I = \langle 0 \rangle \rightsquigarrow V(I(\mathcal{O}(\mathbf{x}_0))) = \mathbb{R}$

Case Study: Longitudinal Dynamics of an Airplane

6th Order Longitudinal Equations

$$\dot{u} = \frac{X}{m} - g\sin(\theta) - qw \qquad u : \text{axial velocity}$$

$$\dot{w} = \frac{Z}{m} + g\cos(\theta) + qu \qquad w : \text{vertical velocity}$$

$$\dot{x} = \cos(\theta)u + \sin(\theta)w \qquad x : \text{range}$$

$$\dot{z} = -\sin(\theta)u + \cos(\theta)w \qquad z : \text{altitude}$$

$$\dot{q} = \frac{M}{I_{yy}} \qquad q : \text{pitch rate}$$

$$\dot{\theta} = q \qquad \theta : \text{pitch angle}$$

Case Study: Generated Invariants

Automatically Generated Invariant Functions

$$\frac{Mz}{I_{yy}} + g\theta + \left(\frac{X}{m} - qw\right)\cos(\theta) + \left(\frac{Z}{m} + qu\right)\sin(\theta)$$

$$\frac{Mx}{I_{yy}} - \left(\frac{Z}{m} + qu\right)\cos(\theta) + \left(\frac{X}{m} - qw\right)\sin(\theta)$$

$$-q^2 + \frac{2M\theta}{I_{yy}}$$

Longitudinal Motion