Characterizing Invariant Algebraic Sets for Polynomial Differential Equations

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Motivations

Qualitative Analysis of Differential Equations is important in its own right!

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Practical Use of Invariant Regions

- Verification and simulation of dynamical and hybrid systems
- Controllers' Synthesis, Stability Analysis
- Numerical Integration, Integrability
- ...

Definitions

Given a polynomial ordinary differential equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$.

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Orbit

$$\mathcal{O}_{\mathbf{x}_0} := \{\mathbf{x}(t) \mid t \in U\} = \{\mathbf{x} \in \mathbb{R}^n \mid \exists t \in \mathbb{R}, \mathbf{x} = \varphi_t(\mathbf{x}_0)\} \subset \mathbb{R}^n$$

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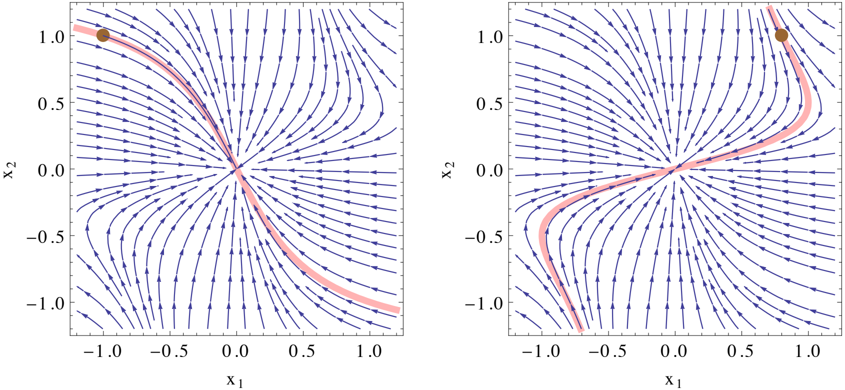
Invariant Region $S \subset \mathbb{R}^n$

$$\forall \mathbf{x}_0 \in S, \forall t \in U, \mathbf{x}(t) \in S$$

Algebraic Invariant Equations

$$\mathbf{f} = (-x_1 - 2x_1^2 x_2, -x_2),$$



$$p(x_1, x_2) = (x_2(0) - x_1(0)x_2(0)^2)x_1 - x_1(0)(x_2 - x_1 x_2^2) = 0$$

More Definitions

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Sets of Common Roots and Vanishing Ideals

$$Y \subset \mathbb{R}[\mathbf{x}],\ V(Y) := \{\mathbf{x} \in \mathbb{R}^n \mid \forall p \in Y, p(\mathbf{x}) = 0\}$$
$$S \subset \mathbb{R}^n,\ I(S) := \{p \in \mathbb{R}[\mathbf{x}] \mid \forall \mathbf{x} \in S, p(\mathbf{x}) = 0\}$$

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Closure (Zariski Topology)

$$\bar{\mathcal{O}}_{\mathbf{x}_0} := V(I(\mathcal{O}_{\mathbf{x}_0}))$$

Outline

Properties of the Zariski Closure

Proposition1: Dimension and Integrability

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$I(\mathcal{O}(\mathbf{x}_0))$ is a (proper) *differential ideal* for $\mathfrak{D}_{\mathbf{f}}$, that is, $\mathfrak{D}_{\mathbf{f}}(p) \in I(\mathcal{O}(\mathbf{x}_0))$ for all $p \in I(\mathcal{O}(\mathbf{x}_0))$

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Example: Zariski Dense Varieties

$\dot{x} = x \rightsquigarrow \mathcal{O}(\mathbf{x}_0) = [0, \infty[\rightsquigarrow I = \langle 0 \rangle \rightsquigarrow \bar{\mathcal{O}}_{\mathbf{x}_0} = V(I(\mathcal{O}(\mathbf{x}_0))) = \mathbb{R}$

Characterizing Elements of $I(\mathcal{O}(\mathbf{x}_0))$

Definition: Differential Order

The *differential order* of $p \in \mathbb{R}[\mathbf{x}]$ denotes the length of the chain of ideals

$$\langle p \rangle \subset \langle p, \mathfrak{D}_{\mathbf{f}}(p) \rangle \subset \cdots \subset \left\langle p, \mathfrak{D}_{\mathbf{f}}(p), \ldots, \mathfrak{D}_{\mathbf{f}}^{(N_p - 1)}(p) \right\rangle =: \partial p.$$

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Theorem

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Proof Sketch

$\leftarrow$: Since $\mathbf{x}(t)$ is analytic, $p(\mathbf{x}(t))$ is also analytic. Thus for a nonempty open neighborhood $V \subset U$ around 0, the null Taylor series of $p(t)$ is equal to p, thus $p = 0$ for all U.

Corollaries

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Corollary2

For each $\mathbf{x}_0$, there exists a unique (up to multiplication by a constant and rearrangement of its factors) $p \in \mathbb{R}[\mathbf{x}]$ such that

$$\partial p = I(\mathcal{O}(\mathbf{x}_0)) \ .$$

Decidability: $\partial p \subset I(V(\langle p \rangle))$

Given $\mathbf{f}$ and $p \in \mathbb{R}[\mathbf{x}]$, the invariance of $V(\langle p \rangle)$ is decidable.

$$\mathfrak{D}_{\mathbf{f}}^{(N_p)}(p) = \sum_{i=0}^{N_p-1} \lambda_i \mathfrak{D}_{\mathbf{f}}^{(i)}(p) \; (\lambda_i \in \mathbb{R}[\mathbf{x}]) \; \wedge \; p = 0 \rightarrow \bigwedge_{i=1}^{N_p-1} \mathfrak{D}_{\mathbf{f}}^{(i)}(p) = 0$$

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$$\mathfrak{D}_{\mathbf{f}}^{(3)}(p) = \sum_{i=0}^{2} \lambda_i \mathfrak{D}_{\mathbf{f}}^{(i)}(p) \; (\lambda_i \in \mathbb{R}[\mathbf{x}]) \; \wedge \; p = 0 \rightarrow \bigwedge_{i=1}^{2} \mathfrak{D}_{\mathbf{f}}^{(i)}(p) = 0$$

$$\mathfrak{D}_{\mathbf{f}}^{(2)}(p) = \lambda_0 p + \lambda_1 \mathfrak{D}_{\mathbf{f}}(p) \; (\lambda_i \in \mathbb{R}[\mathbf{x}]) \; \wedge \; p = 0 \rightarrow \mathfrak{D}_{\mathbf{f}}(p) = 0$$

$$\mathfrak{D}_{\mathbf{f}}(p) = \lambda p \; (\lambda \in \mathbb{R}[\mathbf{x}])$$

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- Existence of λ_i: Gröbner Basis
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Remark: Generalization to a conjunction

Theorem (bis)

The polynomials $p_1, \ldots, p_m$ are in $I(\mathcal{O}(\mathbf{x}_0))$ if and only if $\mathfrak{D}_{\mathbf{f}}^{(i)}(p_j)(\mathbf{x}_0) = 0$, for all $i = 0, \ldots, N-1$ and $j = 1, \ldots, m$, where N is the differential order of the ideal $\langle p_1, \ldots, p_m \rangle$ and satisfies $N \leq \max_j N_{p_j}$.

Practical Use

Over the reals, although

$$\bigwedge_1^m p_j = 0 \quad \equiv_{\mathbb{R}} \quad p := \sum_1^m p_j^2 = 0,$$

so one could use the Theorem as stated for one polynomial p, the SoS increases the complexity of the underlying algorithms.

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Invariant Algebraic Set

Let $J := \left\langle p, \ldots, \mathfrak{D}_{\mathbf{f}}^{(N-1)}(p) \right\rangle$, then

$$\partial J = J \subset I(V(J)) \ .$$

By Corollary1, $V(J)$ is invariant.

First Integrals vs. Limit Cycles

Case 1: Polynomial First Integral

The ideal J is parametrized by $\mathbf{x}_0$.
For all $\mathbf{x}_0 \in \mathbb{R}^n$, $p(\mathbf{x}_0) = 0 \wedge \cdots \wedge \mathfrak{D}_{\mathbf{f}}^{(N-1)}(p)(\mathbf{x}_0) = 0$

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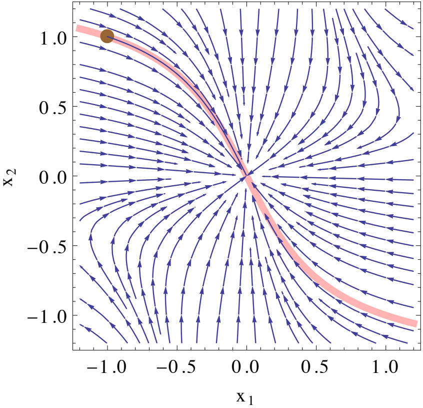
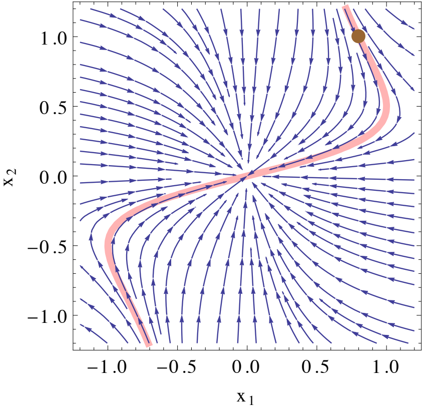
Case 2: Local Invariant Regions (e.g. limit cycle, equilibria)

Restrict $\mathbf{x}_0$ to J, that is

$$p(\mathbf{x}_0) = 0 \wedge \cdots \wedge \mathfrak{D}_{\mathbf{f}}^{(N-1)}(p)(\mathbf{x}_0) = 0$$

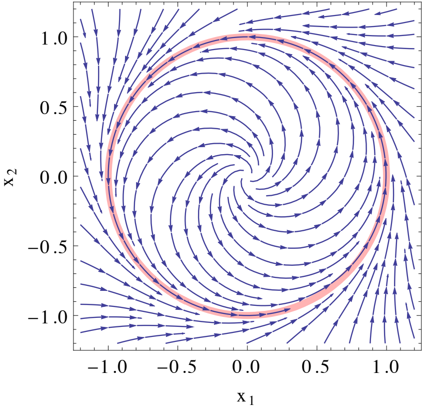
Example: First Integrals



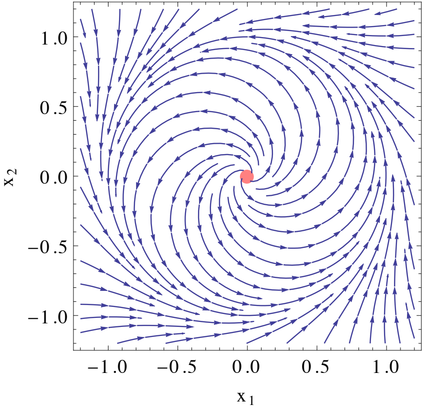


$$p_{(x_1(0),x_2(0))}(x_1, x_2) = (x_2(0) - x_1(0)x_2(0)^2)x_1 - x_1(0)(x_2 - x_1 x_2^2)$$

Example: Local invariant regions



$$p(x_1, x_2) = x_1^2 + x_2^2 - 1$$



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Matrix Representation: Intuition

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$$p = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3$$

$$\begin{array}{ll} (-1+\beta)\alpha_1 & = 0 \\ (-1+\beta)\alpha_2 & = 0 \\ (\beta)\alpha_3 & = 0 \end{array} \leftrightarrow \begin{pmatrix} -1+\beta & 0 & 0 \\ 0 & -1+\beta & 0 \\ 0 & 0 & \beta \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = 0$$

Study the **null space** (kernel) of $M(\beta)$

- Max dim of ker of $M(\beta) \rightsquigarrow$ more “freedom” for $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$
- Increases the chances of finding **first integrals**
- Dually, minimize the rank of $M(\beta) \rightsquigarrow$ **NP-hard** [Buss et al. 1999]

$$h = x_2(0)x_1 - x_1(0)x_2$$

Toward a Generation Procedure ?

We started with a parametrized polynomial p of degree 1 and $N = 1 \ldots$

If no invariants:

- Any reasonable bound on N ? [GM Socias 1992]
- Any bound on the degree of p ?
- Increase order N versus increase the polynomial degree of p ?

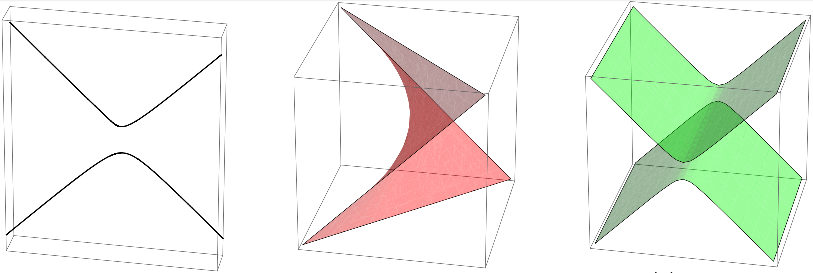
Example1

ODE

$$\dot{x_1} = -x_2 \qquad \dot{x_2} = x_1$$
$$\dot{x_3} = x_4^2 \qquad \dot{x_4} = x_3 x_4$$

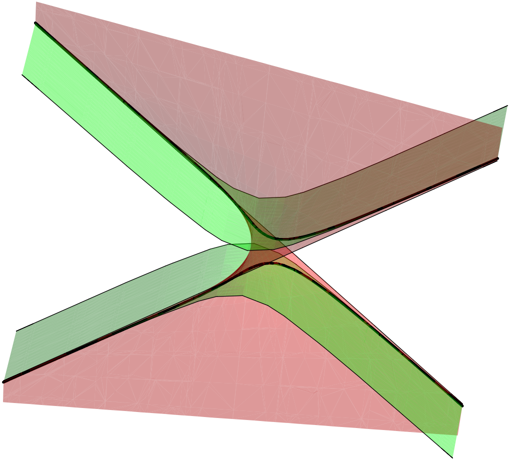
Invariant

$$J = \left\langle p, \mathfrak{D}_{\mathbf{f}}(p), \mathfrak{D}_{\mathbf{f}}^{(2)}(p) \right\rangle = \langle -1 + x_1 x_4,\ x_3 - x_2 x_4,\ x_4^2 - x_3^2 - 1 \rangle$$



Orbit $\qquad \mathfrak{D}_{\mathbf{f}}(p) = 0 \qquad \mathfrak{D}_{\mathbf{f}}^{(2)}(p) = 0$

Example1: cont'd



Example2: Longitudinal Dynamics of an Airplane

6th Order Longitudinal Equations

$\dot{u} = \frac{X}{m} - g\sin(\theta) - qw$	u : axial velocity
$\dot{w} = \frac{Z}{m} + g\cos(\theta) + qu$	w : vertical velocity
$\dot{x} = \cos(\theta)u + \sin(\theta)w$	x : range
$\dot{z} = -\sin(\theta)u + \cos(\theta)w$	z : altitude
$\dot{q} = \frac{M}{I_{yy}}$	q : pitch rate
$\dot{\theta} = q$	θ : pitch angle

Case Study: Generated Invariants

Automatically Generated Invariant Functions

$$\frac{Mz}{I_{yy}} + g\theta + \left(\frac{X}{m} - qw\right)\cos(\theta) + \left(\frac{Z}{m} + qu\right)\sin(\theta)$$

$$\frac{Mx}{I_{yy}} - \left(\frac{Z}{m} + qu\right)\cos(\theta) + \left(\frac{X}{m} - qw\right)\sin(\theta)$$

$$-q^2 + \frac{2M\theta}{I_{yy}}$$

Outline

Related and On Going Work

Absolutely non-exhaustive (Sorry !)

- Liouville, Darboux, Poincaré, Painlevé (Qualitative Analysis)
- Lie, Vessiot, Picard (Differential Galois Theory)
- Ritt, Kolchin (Differential Algebra)
- Prelle, Singer, Ulmer, Weil, Chèze (Effective Methods)
- Basu, Roy, Collins (Real AG, QE)

On Going Work

- Better understand the link between Diff Galois Groups and Invariant Algebraic Sets
- Extensions to Non-Smooth Dynamics (Switched Systems)
- Extension to Differential Algebraic Equations (DAE)

Optional: Differential Algebra

Extension to Differential Algebraic Equations

- ∂ denotes the trivial derivation on R
- define $R\{\mathbf{x}\}$, the ring of differential polynomials ($\partial x_i := x_i'$)
- define $\partial(x_1' - f_1, \ldots, x_n' - f_n)$ as the differential ideal of $R\{\mathbf{x}\}$ generated by all the $x_i' - f_i$
- ∂ could be extended uniquely on $R\{\mathbf{x}\}/\partial(\mathbf{x}' - \mathbf{f})$
- the extension of ∂ on the quotient ring gives an algebraic definition of the Lie derivation
- can we enumerate all differential ideals of the quotient ring ?
- is it a simple differential ring ? (no invariant algebraic sets)