Characterizing Algebraic Invariants by Differential Radical Invariants

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Context: Hybrid Systems Model

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Sensing:  read data from sensors
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Control:  actuate
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Plant:  evolve
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Context: Hybrid Systems Model

Init

$\longrightarrow$

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  Plant: evolve)*

Safety

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$\longrightarrow$

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(Sensing: read data from sensors

Control: actuate

Plant: Evolve)*

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Safety

Evolution

- Continuous time
- Ordinary Differential Equations (ODE)

Algebraic Differential Equations

Example

$$\dot{x_1} = -x_2 \qquad \dot{x_2} = x_1$$
$$\dot{x_3} = x_4^2 \qquad \dot{x_4} = x_3 x_4$$

Formally

$$\frac{dx_i(t)}{dt} = \dot{x}_i = p_i(\mathbf{x}), 1 \leq i \leq n \ .$$

Algebraic Invariant Expression

$$\dot{x_1} = -x_2 \qquad \dot{x_2} = x_1$$

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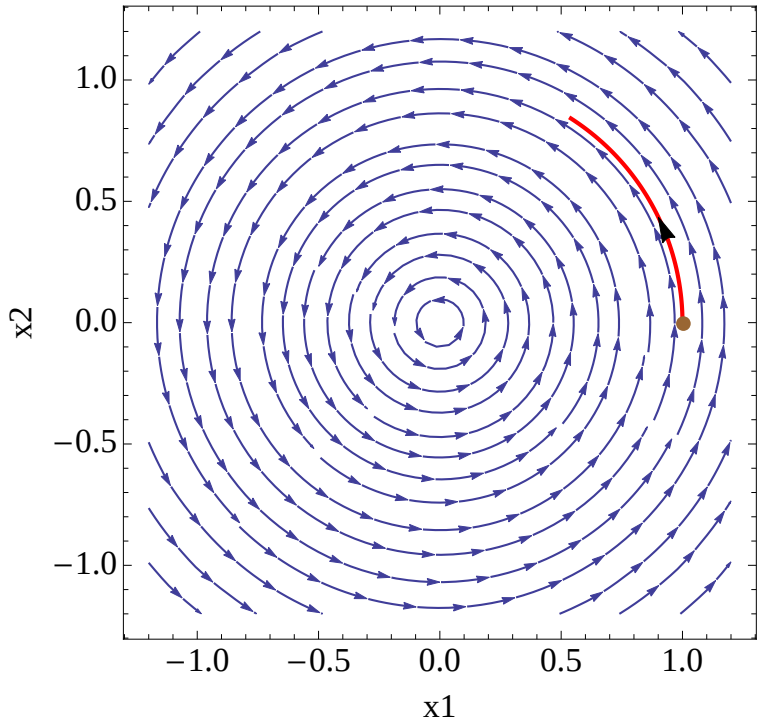

Solution for $\mathbf{x}_0 = (1, 0, 0, 1)$ for $t = [0, 1]$

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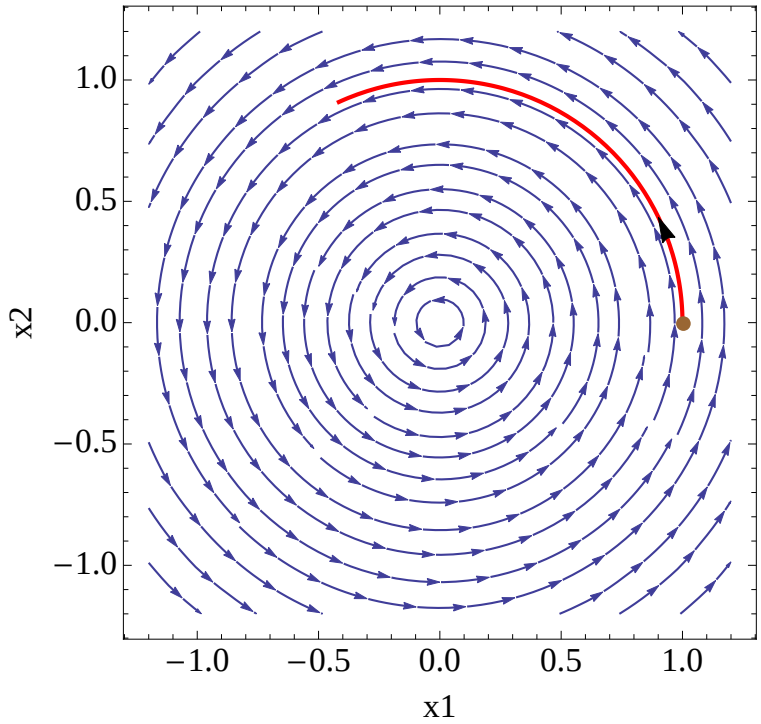

Solution for $\mathbf{x}_0 = (1, 0, 0, 1)$ for $t = [0, 2]$

Algebraic Invariant Expression

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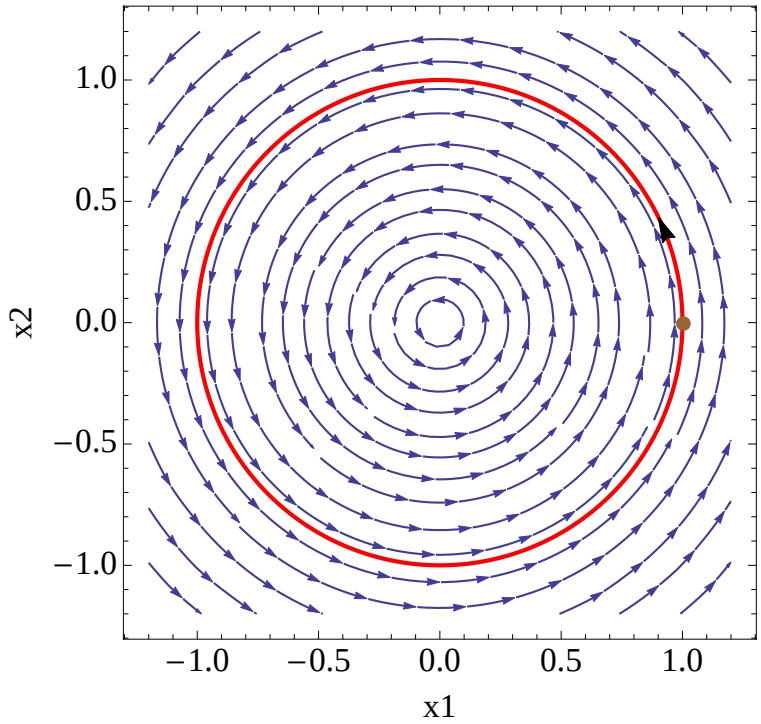

$$\forall t, x_1(t)^2 + x_2(t)^2 - 1 = 0$$

Algebraic Invariant Expression

Geometrically

$\mathcal{O}(\mathbf{x}_0) \stackrel{\text{def}}{=} \{\mathbf{x}(t) \mid t \geq 0\} \in$ Set of roots of $h(x_1, x_2)$

Algebraic Invariant Expression

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$$\mathcal{O}(\mathbf{x}_0) \stackrel{\text{def}}{=} \{\mathbf{x}(t) \mid t \geq 0\} \in \text{Set of roots of } h(x_1, x_2)$$

Algebraically

$$h(x_1, x_2) = x_1^2 + x_2^2 - 1 (= 0)$$

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Algebraically

$h(x_1, x_2) = x_1^2 + x_2^2 - 1 (= 0)$

Algebraic Invariant Expression

$$\forall t, h(\mathbf{x}(t)) = 0,$$

for all $\mathbf{x}(t)$ solution of the Initial Value Problem.

Problems

I. Checking the invariance of Algebraic Expression

Given $\mathbf{p}$, and $\mathbf{x}_0$ root of $h(\mathbf{x})$ ($h(\mathbf{x}_0) = 0$), is $h(\mathbf{x}) = 0$ is an algebraic invariant expression ? ($\forall t \geq 0$, $h(\mathbf{x}(t)) = 0$)

II. Generate Algebraic Invariant Expression

Given $\mathbf{p}$, and `Init`, how to generate algebraic invariant expressions ? ($h(\mathbf{x}) = 0$)

Outline

Variety Embedding of Orbits

Zariski Closure

Vanishing Ideal: all polynomials that vanish on $\mathcal{O}(\mathbf{x}_0)$

$$I(\mathcal{O}(\mathbf{x}_0)) \stackrel{\text{def}}{=} \{h \in \mathbb{R}[\mathbf{x}] \mid \forall \mathbf{x} \in \mathcal{O}(\mathbf{x}_0), h(\mathbf{x}) = 0\}$$

Affine (Real) Variety: common roots of all polynomials in I

$$V(I) \stackrel{\text{def}}{=} \{\mathbf{x} \in \mathbb{R}^n \mid \forall h \in I, h(\mathbf{x}) = 0\}$$

Closure: Sound Abstraction

$$\mathcal{O}(\mathbf{x}_0) \subseteq V(I(\mathcal{O}(\mathbf{x}_0)))$$

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Closure: Sound Abstraction

$$\mathcal{O}(\mathbf{x}_0) \subseteq V(I(\mathcal{O}(\mathbf{x}_0)))$$

$V(I(\mathcal{O}(\mathbf{x}_0)))$ is the **smallest variety** that **contains** $\mathcal{O}(\mathbf{x}_0)$

Variety Embedding

$$\underset{\text{Orbit}}{\mathcal{O}(\mathbf{x}_0)} \xrightarrow{\alpha} \underset{\text{Vanishing Ideal}}{I(\mathcal{O}(\mathbf{x}_0))} \xrightarrow{\gamma} \underset{\text{Closure}}{V(I(\mathcal{O}(\mathbf{x}_0)))} \supseteq \underset{\text{Orbit}}{\mathcal{O}(\mathbf{x}_0)}$$

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Limitation: Zariski Dense Varieties

$\dot{x} = x \rightsquigarrow \mathcal{O}(\mathbf{x}_0) = [\mathbf{x}_0, \infty[\rightsquigarrow I = \langle 0 \rangle \rightsquigarrow V(I(\mathcal{O}(\mathbf{x}_0))) = \mathbb{R}$

Outline

Properties of $I(\mathcal{O}(\mathbf{x}_0))$ (abstract domain)

$I(\mathcal{O}(\mathbf{x}_0))$: The set of all polynomials that vanish on $\mathcal{O}(\mathbf{x}_0)$

$I(\mathcal{O}(\mathbf{x}_0))$ is an ideal

- $0 \in I(\mathcal{O}(\mathbf{x}_0))$
- if $h_1, h_2 \in I(\mathcal{O}(\mathbf{x}_0))$, then $h_1 + h_2 \in I(\mathcal{O}(\mathbf{x}_0))$
- if $h \in I(\mathcal{O}(\mathbf{x}_0))$, then $q\,h \in I(\mathcal{O}(\mathbf{x}_0))$, for any polynomial q

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$I(\mathcal{O}(\mathbf{x}_0))$ is a Differential Ideal

if $h \in I(\mathcal{O}(\mathbf{x}_0))$, then $\frac{dh}{dt} \in I(\mathcal{O}(\mathbf{x}_0))$.

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Instead of $\frac{dh}{dt}$, we will use $\mathfrak{L}_{\mathbf{p}}(h)$: The **Lie Derivative** of h.

Lie derivative along a vector field

Definition

$$\mathfrak{L}_{\mathbf{p}}(h) \stackrel{\text{def}}{=} \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} p_i(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]$$

Properties

- Algebraic differentiation (chain rule)
- Do not require $\mathbf{x}(t)$ (the solution of the ODE)
- Corresponds to the time derivative when $\mathbf{x} = \mathbf{x}(t)$

Differential Radical Invariants

Theorem

$h \in I(\mathcal{O}(\mathbf{x}_0))$ **if and only if**

- $\exists\, g_0, \ldots, g_{N-1} \in \mathbb{R}[\mathbf{x}]$: $\mathfrak{L}_{\mathbf{p}}^{(N)}(h) = \sum_{i=0}^{N-1} g_i \mathfrak{L}_{\mathbf{p}}^{(i)}(h)$
- N **finite**
- $\mathfrak{L}_{\mathbf{p}}^{(0)}(h)(\mathbf{x}_0) = 0, \ldots, \mathfrak{L}_{\mathbf{p}}^{(N-1)}(h)(\mathbf{x}_0) = 0$

Special Cases

Invariant Polynomial Functions

$\mathfrak{L}_{\mathbf{p}}(h) = 0 \longleftrightarrow N = 1$ and $g_0 = 0$

Darboux Polynomials [Darboux 1878, Painlevé, Poincaré . . .]

$\mathfrak{L}_{\mathbf{p}}(h) = g_0\, h \longleftrightarrow N = 1$

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Outline

Checking Invariance of Candidates

I. Checking the invariance of Algebraic Expression

Given $\mathbf{p}$, and $\mathbf{x}_0$ root of $h(\mathbf{x})$ ($h(\mathbf{x}_0) = 0$), is $h(\mathbf{x}) = 0$ is an algebraic invariant expression ? ($\forall t \geq 0,\ h(\mathbf{x}(t)) = 0$)

A Necessary and Sufficient Proof Rule

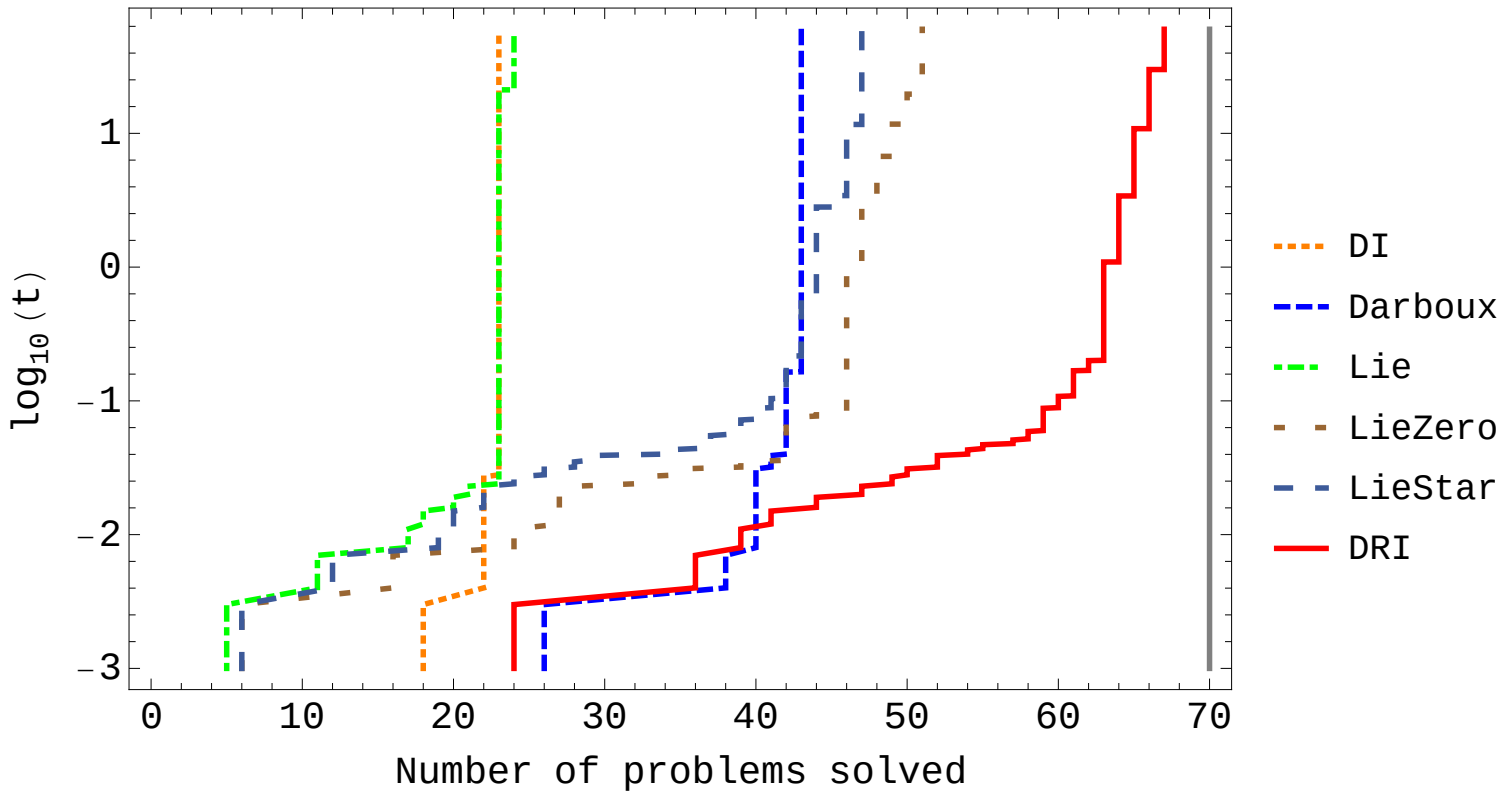
$$(\text{DRI})\ \frac{h = 0 \rightarrow \bigwedge_{i=0}^{N-1} \mathfrak{L}_{\mathbf{p}}^{(i)}(h) = 0}{(h = 0) \rightarrow [\dot{\mathbf{x}} = \mathbf{p}](h = 0)}\ .$$

DRI: Algorithm

Data: h, **p**, **x**
Result: Boolean: True, False.
$N \leftarrow 1$
$\texttt{GB} \leftarrow \{h\}$
$\ell \leftarrow h$
$\texttt{symbs} \leftarrow \text{Variables}[\mathbf{p}, h]$
while true **do**
  $\texttt{GB} \leftarrow \text{GröbnerBasis}[\texttt{GB}, \mathbf{x}]$
  $\texttt{LieD} \leftarrow \text{LieDerivative}[\ell, \mathbf{p}, \mathbf{x}]$
  $\texttt{Rem} \leftarrow \text{PolynomialRemainder}[\texttt{LieD}, \texttt{GB}, \mathbf{x}]$
  if $\texttt{Rem} = 0$ **then**
    return True
  else
    $\texttt{Reduce} \leftarrow \text{QE}[\forall\ \texttt{symbs}, h = 0 \rightarrow \texttt{LieD} = 0]$
    if $\neg\texttt{Reduce}$
    then
      return False
    Append[GB, LieD]
    $\ell \leftarrow \texttt{LieD}$
    $N \leftarrow N + 1$

Benchmarks

Joint work with Andrew Sogokon



Generation of Invariant Varieties

II. Generate Algebraic Invariant Expression

Given $\mathbf{p}$, and `Init`, how to generate algebraic invariant expressions ? ($h(\mathbf{x}) = 0$)

Theorem

$S \subseteq \mathbb{R}^n$ is an invariant variety **if and only if** S is the set of roots of the system

$$\bigwedge_{0 \leq i \leq N-1} \mathfrak{L}_{\mathbf{p}}^{(i)}(h) = 0,$$

for some polynomial h.

In practice

In practice

Find h and N, such that:

$$\mathfrak{L}_{\mathbf{p}}^{(N)}(h) = \sum_{i=0}^{N-1} g_i \mathfrak{L}_{\mathbf{p}}^{(i)}(h)$$

➔ The set of roots of

$$\bigwedge_{0 \leq i \leq N-1} \mathfrak{L}_{\mathbf{p}}^{(i)}(h) = 0,$$

is an invariant variety.

Matrix Representation: Intuition

invariant of degree 1

$$\dot{x_1} = a_1x_1 + a_2x_2 \qquad h = \alpha_1x_1 + \alpha_2x_2 + \alpha_3x_0$$
$$\dot{x_2} = b_1x_1 + b_2x_2 \qquad \mathfrak{L}_{\mathbf{p}}(h) = \alpha_1(a_1x_1 + a_2x_2) + \alpha_2(b_1x_1 + b_2x_2)$$

$$\exists ?\beta \in \mathbb{R} \text{ s.t. } \mathfrak{L}_{\mathbf{p}}(h) = \beta h$$

$$\begin{array}{rl} (-a_1+\beta)\alpha_1 + (-b_1)\alpha_2 & = 0 \\ (-a_2)\alpha_1 + (-b_2+\beta)\alpha_2 & = 0 \\ (\beta)\alpha_3 & = 0 \end{array} \leftrightarrow \begin{pmatrix} -a_1+\beta & -b_1 & 0 \\ -a_2 & -b_2+\beta & 0 \\ 0 & 0 & \beta \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = 0$$

Matrix Representation

Polynomial	$\leftrightarrow$	$\binom{n+d}{d}$ Coefficients (d degree of h)
h	$\leftrightarrow$	$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_r)$
g_i	$\leftrightarrow$	$\boldsymbol{\beta}_i = (\beta_1, \beta_2, \ldots, \beta_{s_i})$

Matrix Representation

$$\mathfrak{L}_{\mathbf{p}}^{(N)}(h) = \sum_{i=0}^{N-1} g_i \mathfrak{L}_{\mathbf{p}}^{(i)}(h) \;\leftrightarrow\; M(\boldsymbol{\beta})\boldsymbol{\alpha} = 0$$

$\boldsymbol{\alpha}$ lies in the **Kernel** of $M(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \{\boldsymbol{\alpha} \in \mathbb{R}^r \mid M(\boldsymbol{\beta})\boldsymbol{\alpha} = 0\}$

Example: $n = 2,\ d = 1,\ N = 1$

invariant of degree 1

$$\dot{x_1} = a_1 x_1 + a_2 x_2 \qquad h = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_0$$
$$\dot{x_2} = b_1 x_1 + b_2 x_2 \qquad \mathfrak{L}_{\mathbf{p}}(h) = \alpha_1(a_1 x_1 + a_2 x_2) + \alpha_2(b_1 x_1 + b_2 x_2)$$

$$|M(\boldsymbol{\beta})| = \beta(\beta^2 - (a_1 + b_2)\beta - a_2 b_1 + a_1 b_2)$$

- $\ker(M(0)) = \langle (0,0,1) \rangle$
- $\boldsymbol{\alpha} \in \langle (0,0,1) \rangle \cap \mathbf{x}_0{}^{\perp}$

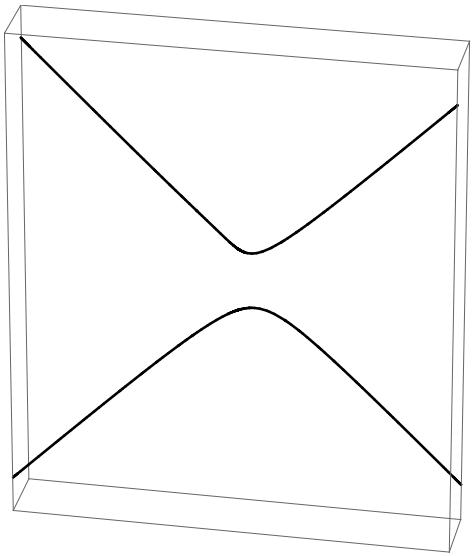
Example

System

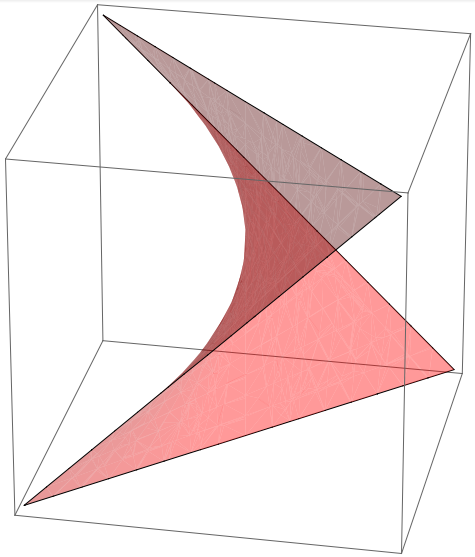
$$\dot{x_1} = -x_2 \qquad \dot{x_2} = x_1$$
$$\dot{x_3} = x_4^2 \qquad \dot{x_4} = x_3 x_4$$

Differential Radical Invariants

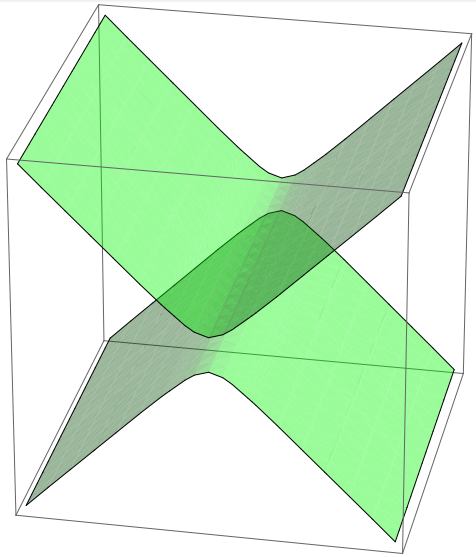
$h = -1 + x_1 x_4$ and $\mathfrak{L}_{\mathbf{p}}(h) = x_3 - x_2 x_4$ and $\mathfrak{L}_{\mathbf{p}}^{(2)}(h) = x_4^2 - x_3^2 - 1$



Orbit

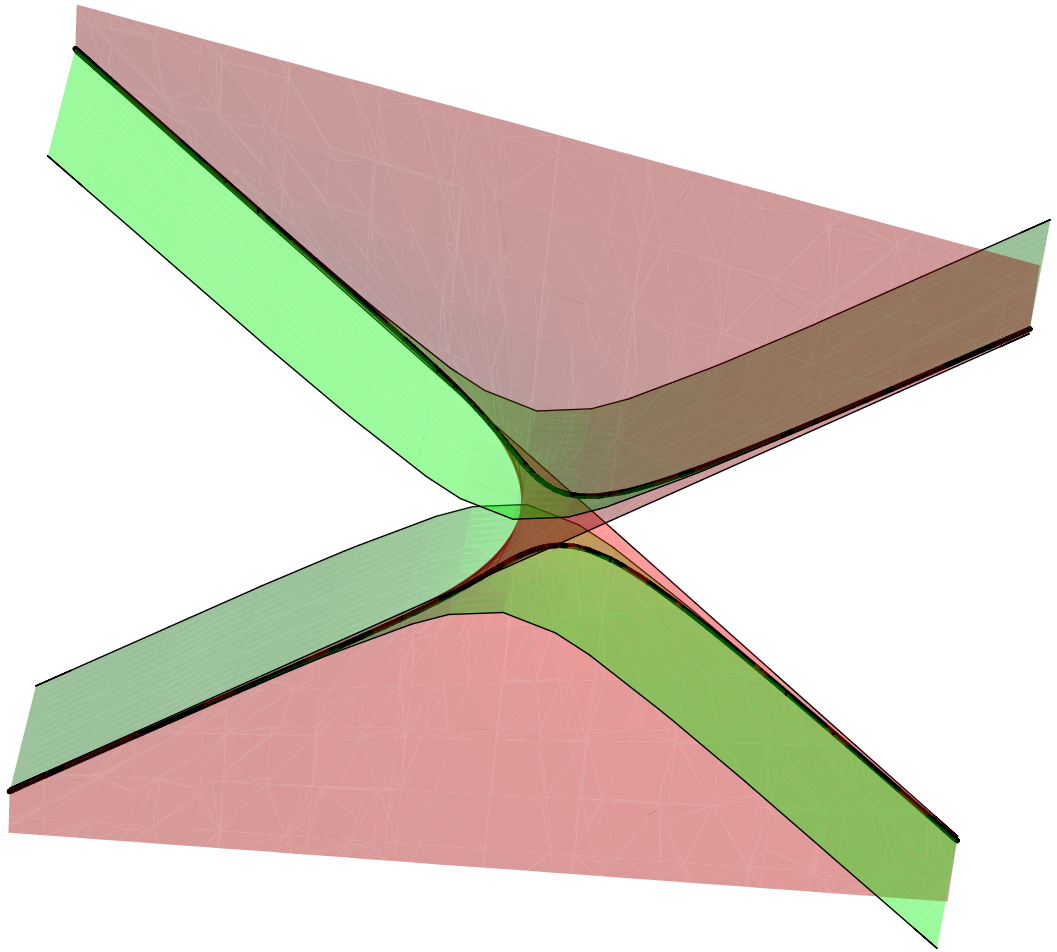


Roots of $\mathfrak{L}_{\mathbf{p}}(h)$



Roots of $\mathfrak{L}_{\mathbf{p}}^{(2)}(h)$

Example: cont'd



Overapproximation of $\mathcal{O}(\mathbf{x}_0)$

Case Study: Longitudinal Dynamics of an Airplane

6th Order Longitudinal Equations

$$\dot{u} = \frac{X}{m} - g\sin(\theta) - qw \qquad u : \text{axial velocity}$$

$$\dot{w} = \frac{Z}{m} + g\cos(\theta) + qu \qquad w : \text{vertical velocity}$$

$$\dot{x} = \cos(\theta)u + \sin(\theta)w \qquad x : \text{range}$$

$$\dot{z} = -\sin(\theta)u + \cos(\theta)w \qquad z : \text{altitude}$$

$$\dot{q} = \frac{M}{I_{yy}} \qquad q : \text{pitch rate}$$

$$\dot{\theta} = q \qquad \theta : \text{pitch angle}$$

Case Study: Generated Invariants

Automatically Generated Invariant Functions

$$\frac{Mz}{I_{yy}} + g\theta + \left(\frac{X}{m} - qw\right)\cos(\theta) + \left(\frac{Z}{m} + qu\right)\sin(\theta)$$

$$\frac{Mx}{I_{yy}} - \left(\frac{Z}{m} + qu\right)\cos(\theta) + \left(\frac{X}{m} - qw\right)\sin(\theta)$$

$$-q^2 + \frac{2M\theta}{I_{yy}}$$

Conclusion

- Variety embedding of the reachable set (Zariski Closure)
- Characterizing elements of the vanishing ideal $I(\mathcal{O}(\mathbf{x}_0))$
- New Necessary and sufficient proof rule (DRI)
- Leveraging linear algebra tools to generate algebraic invariant equations

Thank you for attending !

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Enforcing Invariants

$\delta \stackrel{\text{def}}{=} (a_1 - b_2)^2 + 4a_2b_1 \geq 0$

- $\beta \in \{0, \frac{1}{2}(a_1 + b_2 + \sqrt{\delta}), \frac{1}{2}(a_1 + b_2 - \sqrt{\delta})\}$
- If $\mathbf{x}_0 \in \left(a_1 - b_2 \pm \sqrt{\delta}, 2a_2, 0\right)^{\perp}$ then $\boldsymbol{\alpha} = \left(a_1 - b_2 \pm \sqrt{\delta}, 2a_2, 0\right)$
- $\boldsymbol{\alpha}$ is an **Eigenvector**

If $a_2 = 0$, $\beta \in \{0, a_1, b_2\} \rightsquigarrow \ldots$