The Shimizu–Morioka System has no Nontrivial Darboux Polynomials

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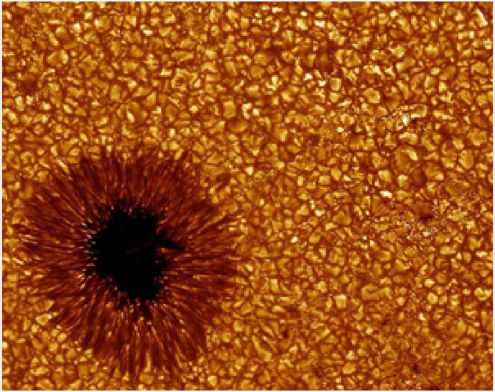
on July 15th, AADIOS 2025, Heraklion, Greece

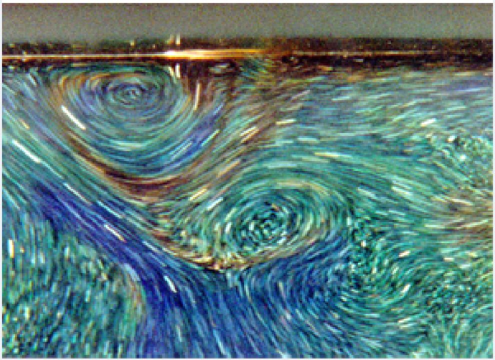
Joint work with Maxime Bridoux (Inria, France)

» Turbulent Convection



Granules and a sunspot in the sun's photosphere observed on 8 August 2003 by Göran Scharmer and Kai Langhans with the Swedish 1-m Solar Telescope. (`https://physics.aps.org/articles/v2/74`)

» Lab Experiments



Cooler regions appear brown and warmer regions appear green and blue. This image was taken near the top surface. (...) one sees a brownish (cold) plume detaching from the boundary layer, extending down and to the left into the fluid, and forming a mushroom head consisting of two swirls. (`https://physics.aps.org/articles/v2/74`)

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Physics Letters A

Volume 76, Issues 3–4, 31 March 1980, Pages 201-204

On the bifurcation of a symmetric limit cycle to an asymmetric one in a simple model

T. Shimizu, N. Morioka

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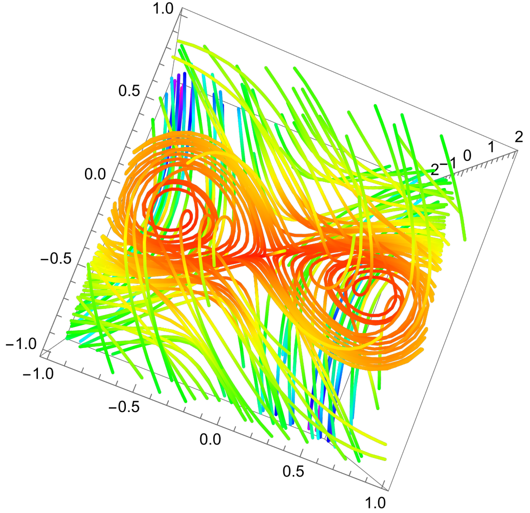
https://doi.org/10.1016/0375-9601(80)90466-1 ↗

Abstract

A simple model, the solution of which shows a similar bifurcation as in the Lorenz model for high Rayleigh numbers, is proposed. The analytic form of the limit cycles is calculated by using perturbation theory. The bifurcation is discussed.

» Shimizu-Morioka System

$$\dot{x} = y$$
$$\dot{y} = x - \lambda y - xz$$
$$\dot{z} = -\alpha z + x^2$$



$\alpha = 0.45, \lambda = 0.75$

» Integrability Analysis (2020)

K. Huang, S. Shi, and W. Li

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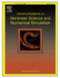
Communications in Nonlinear Science and Numerical Simulation

Volume 84, May 2020, 105101



Research paper

Integrability analysis of the Shimizu–Morioka system

Kaiyin Huang [a], Shaoyun Shi [b c], Wenlei Li [b]

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https://doi.org/10.1016/j.cnsns.2019.105101

Highlights

- We show that a homothetic transformation converts the Shimizu–Morioka system into the Rucklidge system.
- The results on the Shimizu–Morioka system can be obtained from the corresponding results on the Rucklidge system.
- Integrability analysis of the Shimizu–Morioka system is performed.

» Integrability by Darboux Polynomials

In Section 3, we firstly try to find all polynomial, rational first integrals of the Shimizu–Morioka system with the help of Darboux theory of integrability: when $\alpha \neq 0$, Darboux integrability of the Shimizu–Morioka system can be trivially derived from the corresponding results on the Rucklidge system; when $\alpha = 0$, we investigate the Darboux first integrals of the Shimizu–Morioka system using Gröbner basis in algebraic geometry. We also prove that there is no any global C^1 first integral under convenient assumptions through the stability or instability of the singular points and periodic orbits.

In Section 4, from the nonintegrability point of view, we show that the Shimizu–Morioka system with $\alpha \neq 0$ is rationally nonintegrable in the sense of Bogoyavlenskij for almost all parameter values, and the Shimizu–Morioka system with $\alpha = 0$ is not algebraically integrable, respectively.

» Conjecture

Remark 5 *We point out that Theorem 4 does not answer the question:* " Is there Darbux polynomials with degree more than three for system (1) with $\alpha = 0$?" *because we do not know if it has Darboux polynomials of degree large than three. However, each irreducible factors of a Darboux polynomial is also a Darboux polynomial, and Darboux polynomials found in applications [22–25] usually linear, quadratic or cubic polynomials. Thus, we conjecture that* the Shimizu-Morioka model has no any Darboux polynomials when $\alpha = 0$.

» Darboux Polynomials (1878)

Given a polynomial ODE

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, \ldots, x_n) \\ &\vdots \\ \dot{x}_n &= f_n(x_1, \ldots, x_n) \end{aligned}$$

a polynomial p is Darboux if and only if

$$\dot{p} \triangleq f \cdot \nabla p = q\, p$$

for some polynomial q, called the cofactor of p

» Generic Polynomials

Generic Monomial

$$x^{d+\gamma} = x^d \quad x^\gamma = x_1^{d_1} \cdots x_n^{d_n} \quad x_1^{\gamma_1} \cdots x_n^{\gamma_n}$$

where

* $\gamma \in \mathbb{Z}^n$
* $d = (d_1, \ldots, d_n)$ is a symbolic vector

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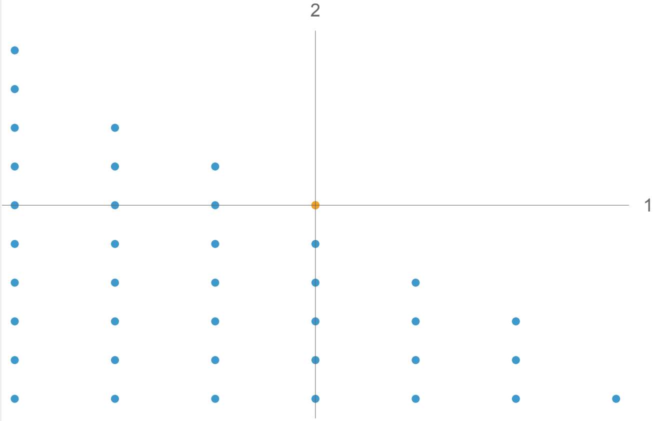
Generic Polynomial

$$\sum_{\gamma \in \Gamma_d} p_{d+\gamma} \; x^{d+\gamma}$$

where

$$\Gamma_d := \{\gamma \in \mathbb{Z}^n | \gamma \preceq 0 \; \wedge \; \gamma + d \geq 0\}$$

» A Generic Polynomial



Rationally independent weights $(\sqrt{2}, 1)$

» Differentiation and Reduction

Differentiation

Let ∂ denote a polynomial derivation and p a generic polynomial. Then ∂p is a generic polynomial.

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Reduction

There exists a unique standard polynomial q such that $\partial p = qp + r$ where r is a generic polynomial and none of its (generic) monomials is *divisible* by x^d.

» Cofactor Support

Tight Over-approximation

Let ∂ denote a derivation and let p denote an eigenpolynomial for ∂. Then either $g = 0$ or

$$\mathrm{Supp}(g) \subseteq \mathcal{D} \cap \mathbb{N}^n$$

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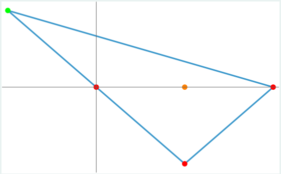
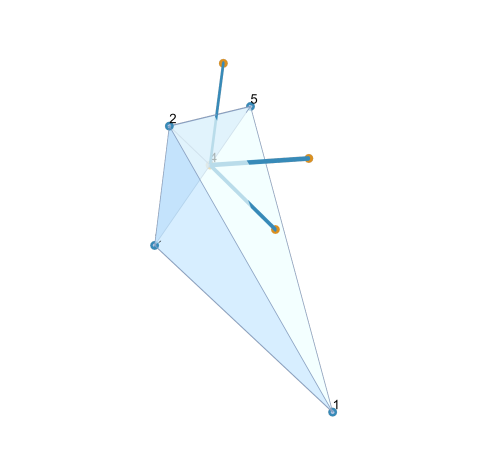
Example

Van Der Pol Oscillator



$$\dot{x} = y, \dot{y} = \mu(1 - x^2)y - x$$



Derivation Polytope

» Necessary Conditions

Coefficients of the Remainder

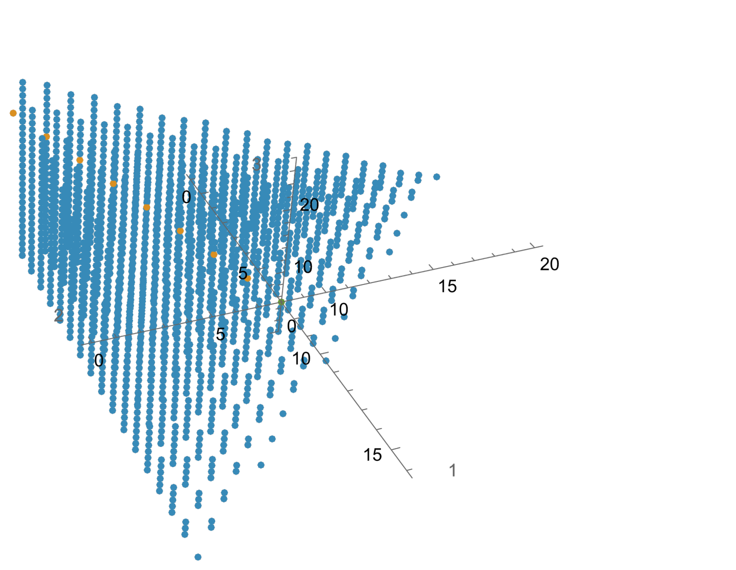
Let ∂ denote a polynomial derivation. If the generic polynomial p is an eigenpolynomial with cofactor $g = \sum g_\beta x^\beta$, then, for all $\gamma \in \mathcal{Z}^n$

$$\sum_{\gamma \preceq \sigma} \lambda_{\gamma,\sigma} p_{d+\gamma-\sigma} = 0$$

where σ ranges over the monomials of $\mathcal{D}$ and $\lambda_{\gamma,\sigma}$ is a polynomial in d and the coefficients of g.

» A Generic Polynomial

Plot for $d = (7, 8, 5)$



» Necessary Recurrence Relations

Consider the sequence

$$u_k := p_{d_1+2k, d_2-2k, d_3+k}, \qquad k = 0, 1, \ldots$$

Then we get the following recurrence relations

$$\begin{aligned}
0 &= 2u_1 - (0 + d_2)u_0 \\
0 &= 4u_2 - (-2 + d_2)u_1 \\
0 &= 6u_3 - (-4 + d_2)u_2 \\
&\vdots \\
0 &= 2ku_k - (-2(k-1) + d_2)u_{k-1}
\end{aligned}$$

» Necessary Recurrence Relations

$w = (1, \sqrt{3}, \sqrt{2})$

Consider the sequence

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Telescoping

$$2(k!)u_k = d_2(d_2 - 2) \cdots (d_2 - 2(k-1))u_0$$

» Necessary Conditions on the Multidegree

Explicit Expression for p_{d_1+2k,d_2-2k,d_3+k} (Falling Factorial)

$$2(k!)p_{d_1+2k,d_2-2k,d_3+k} = d_2(d_2-2)\cdots(d_2-2(k-1))$$

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Suppose d_2 is odd, $d_2 = 2s-1$. Consider $k = s$. Then

* $p_{d_1+2s,d_2-2s,d_3+s} = p_{d_1+2s,-1,d_3+s} = 0$
* $d_2(d_2-2)\cdots(d_2-2(s-1)) = 0$, a contradiction
* Thus d_2 is even

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The coefficient of (d_1-1, d_2+1, d_3) in $-gp + D(p)$ is d_1

* Thus $d_1 = 0$

» Finishing the Proof

Consider the sequence $v_k = p_{d_1+3+2k, d_2-1-2k, d_3-1+k}$

* $(-3)v_0 = d_3$
* Thus, $d_2 = 0$ implies $d_3 = 0$

* $(-5)v_1 = \frac{1}{3}(-1 + d_2)d_3 + \frac{1}{2}d_2(1 + d_3)$
* Thus, $d_2 = 2$ implies $\frac{4}{3}d_3 + 1 = 0$, a contradiction

* $(-7)v_2 = \frac{1}{8}(-2 + d_2)d_2(2 + d_3) - \frac{1}{5}v_1$
* $d_2 = 4$ implies $\frac{12}{5} + \frac{8}{5}d_3 = 0$, a contradiction

* $\cdots$

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- $\cdots$

Theorem

The Shimizu-Morioka model has no non-trivial Darboux polynomial over any field (for all valuations of its parameters).