A Companion Darboux Polynomial for Parametric Linear ODE

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on September 2, 2024

» Darboux Polynomials

Given a polynomial ODE

$$\dot{x}_1 = f_1(x_1, \ldots, x_n)$$
$$\vdots$$
$$\dot{x}_n = f_n(x_1, \ldots, x_n)$$

a polynomial p is Darboux if and only if

$$\dot{p} \triangleq f \cdot \nabla p = q\, p$$

for some polynomial q, called the cofactor of p

» History and Recent Interests

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* Instrumental in integrability theory (1983)
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Open Problems

* Criteria for existence
* Bound on the degree of irreducible Darboux
* Efficient enumeration

» Linear ODE

$$\dot{x} = Ax$$

* Real eigenvalue $\iff$ linear Darboux
* Complex eigenvalue $\iff$ quadratic Darboux
* Rational relations $\iff$ cubic and higher Darboux

» Parametric Linear ODE

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} a_{1,1} & 0 & a_{1,3} \\ 1 & a_{2,2} & a_{2,3} \\ a_{3,1} & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Challenge

Is there an effective way to define a Darboux polynomial for parametric linear ODE?

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tentative

Theorem

Let $\dot{x} = A^t x$ denote a linear ODE. then

$$\delta_A(x) = \det(\pi_A(K(x)))$$

is a Darboux polynomial for the considered ODE.

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$$\delta_A(x) = \det(\pi_A(K(x)))$$

is a Darboux polynomial for the considered ODE.

- $K(x)$ is obtained from A by adding x to its first row
- π_A is the characteristic polynomial of A
- the cofactor of δ_A is $\mathrm{Tr}(A)$ (like the Wronskian)

» Common mistake..

Let $K(x)$ be the matrix obtained by adding x to the first row of A and leaving the other rows unchanged.

It means

$$K(x) = \begin{pmatrix} \mathrm{a11{+}x1} & \mathrm{a12{+}x2} & \mathrm{a13{+}x3} \\ \mathrm{a21} & \mathrm{a22} & \mathrm{a23} \\ \mathrm{a31} & \mathrm{a32} & \mathrm{a33} \end{pmatrix}.$$

It is claimed:

Then $\delta_A(x) := \det(\pi_A(K(x)))$ is a Darboux polynomial of cofactor $tr(A)$ (where $\pi_A(t) = \det(A - tI)$).

But in our case $n = 3$ we have

$$\delta_A(x) := \det(\det(\pi_A(K(x))) = \det(\det(A{-}K(x))) = \det\left(\det\begin{pmatrix} -\mathrm{x1} & -\mathrm{x2} & -\mathrm{x3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) = 0.$$

But it is not a Darboux polynomial!

» Proof sketch

- Let $p(t)$ denote a column of the adjugate matrix of $A - tI$
- Let λ_i denote an eigenvalue of A
- Then $A\, p(\lambda_i) = \lambda_i p(\lambda_i)$

$$\delta_A(x) = \prod_{i=1}^{n} (p(\lambda_i) \cdot x) = \det(\pi_A(K(x)))$$

» Intriguing Property

Let $p(t)$ denote a column of the adjugate matrix of $A - tI$

$$p(t) = C \begin{pmatrix} (-t)^{n-1} \\ \vdots \\ (-t) \\ 1 \end{pmatrix}$$

Conjecture

$\det(C)$ divides all the coefficients of δ_A.

» Objectives

* Remove the dependency to p (prove the conjecture)
* Make δ_A invariant under matrix similarity
* Directly compute the coefficients of δ_A

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Thanks for attending!