On Covering Euclidean Space with Q-arrangements of Cones

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Abstract

This paper is concerned with a covering problem of Euclidean space by a particular arrangement of cones that are not necessarily full and are allowed to overlap. The problem provides an equivalent geometric reformulation of the solvability of the linear complementarity problem defining the class of Q-matrices. Assuming feasibility, we rely on standard tools from convex geometry to study maximal connected uncovered regions, we term *holes*. We then use our approach to fully characterize the problem for dimension 3, regardless of degeneracy. We further provide, for $n \leq 3$, an algebraic characterization for the class of Q-matrices. That is, we show that, M is a Q-matrix if and only if its entries belong to an explicit semi-algebraic set (in dimension 9) where all the involved polynomials are subdeterminants of M. We showcase the usefulness of such a characterization by generating 3-by-3 Q-matrices with specific interesting properties on the involved cones.

Keywords. linear complementarity problem, Q-matrix, covering of Euclidean space, convex geometry, algebraic characterization, symbolic computation.

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Introduction

Given a vector $q \in \mathbb{R}^n$ and an $n \times n$ matrix M over the reals, the *linear complementarity problem*, LCP(q, M), asks whether there exists a pair $w, z \in \mathbb{R}^n$ satisfying $w - Mz = q$, $w, z \geq 0$, and $w.z = 0$, where $w, z \geq 0$ means that w and z belong to $\mathbb{R}^n_+$, the nonnegative orthant of $\mathbb{R}^n$, and $w.z$ is the scalar product of w and z (cf. [Cottle et al., 2009]). When LCP(q, M) admits a solution, it is said to be *solvable*. When a solution satisfying only $w, z \geq 0$ exists (i.e. when dropping the scalar product requirement), LCP(q, M) is said to be *feasible*. Related to the solvability and feasibility concepts, several classes of matrices were defined in the literature. Three classes are in particular relevant to this work. When LCP(q, M) is feasible for all q, M is called an *S-matrix*. When LCP(q, M) is solvable for all q, M is called a *Q-matrix*. If furthermore such a solution is unique for all q, M is called a *P-matrix*.

The solvability of linear complementarity problems was tackled from different angles.
In [Cottle et al., 1981a], a focus on the linear application represented by M proved to be useful. Degree theory [Pang, 1979, Chapter 6] was exploited for certain classes of structured matrices (e.g. [Garcia et al., 1983]), and more recently a new sufficient condition was provided in [Radons and Tonelli-Cueto, 2023]. More in line with this work, [Murty, 1972] pointed out an insightful geometric interpretation for LCP(q, M). Instead of fixing $q \in \mathbb{R}^n$ and solving for w and z, one could instead fix the pair (w, z) and solve for the vectors q. The constraints $w, z \geq 0$ and $w.z = 0$ imply that, for each i, either w_i or z_i has to vanish and the remaining component has to be nonnegative making q an element of a cone spanned by some columns of $-M$ and I (the identity matrix in dimension n). In turn, asking for a solution for each $q \in \mathbb{R}^n$ becomes equivalent to asking whether $\mathbb{R}^n$ is covered by the union of these cones.

In the late fifties, [Samelson et al., 1958] characterized a partition of Euclidean space with 2^n full cones using a separation condition. Their characterization provides a geometric reformulation of what later became known as the class of P-matrices (with the separation being captured as the positiveness of the principal minors of M). Several serious attempts have been subsequently made to characterize the class of Q-matrices concisely and, despite the rich literature devoted to the problem, it remains open even for low dimensions, see e.g. [Fredricksen et al., 1986]. [Kelly and Watson, 1979] proved that, while the set of non-degenerate Q-matrices is open for $n = 3$, it is not for $n = 4$. Some subclasses of Q-matrices with adequate structures were easier to tackle. The problem of recognizing a P-matrix was shown to be co-NP-complete by [Coxson, 1994]. A larger class than P-matrices relies on oriented matroids realized by I and $-M$ was also studied in [Watson, 1974, Morris Jr, 1986]. [Morris Jr, 1988] provided a (counter) example for $n = 4$ showing that the signs of subdeterminants of M alone are not enough to characterize Q-matrices. As of today, the cost of checking generic Q-matrices remains prohibitive in practice, see [Aganagić and Cottle, 1978, Naiman and Stone, 1998], and the best known practical approach presented in [De Loera and Morris Jr., 1999] uses secondary and universal polytopes and is limited to dimensions less than 10. It's worth noting that these theoretical facts about the hardness of the problem are in contrast with the complexity of solving LCP(q, M) for a *fixed* q and M. Indeed, while the general problem (over the integers) has been shown to be NP-complete by [Chung, 1989], there are specific practical instances for which one can go beyond thousands of variables (cf. e.g. [Brugnano and Casulli, 2008]).

The rest of the paper is organized as follows. After a formal introduction of the problem (Section 1), we investigate the relatively simpler feasibility problem to arrive at useful cones we term *minimal* (Section 2). Section 3 focuses on solvability where we revisit the original *separation condition* by [Samelson et al., 1958] in Section 3.1 and show a similar necessary (but not sufficient) condition for Q-matrices. This condition insinuates a lead to refine the standard concept of separation using intersections of some specific cones instead of hyperplanes separating vectors (Sections 3.2 and 3.3). These different ingredients are leveraged in Section 4 to give a complete characterization of Q-matrices in dimension 3 regardless of degeneracy. Finally, section 5 provides an equivalent algebraic characterization in terms of signs of the subdeterminants of the involved matrix.

Contributions. To the best of our knowledge, the following are novel results related to Q-matrices. Proposition 1 gives a necessary condition useful to characterize S-matrices, i.e. the feasibility problem, by triangulating the space. Proposition 4 provides a necessary separation condition for the Q-covering problem (cf. Definition 2). Proposition 5 gives an interesting covering property of particular relevant cones we term *minimal cones*. The remaining results concern only dimension 3. Theorem 1 characterizes uncovered regions assuming feasibility. Corollary 1 strengthens [Garcia et al., 1983, Theorem 4.7] by dropping the strong non-degeneracy assumption. Theorem 2 gives necessary and sufficient conditions for the Q-covering problem. The algorithms in Section 5 provide an explicit algebraic characterization for Q-matrices as sign conditions of the subdeterminants of the involved matrix (Theorem 6). Such a characterization turned out to be very convenient to generate Q-matrices with interesting properties (cf. Examples 1 and 2).

1 Q-covering

Let $g_1, \dots, g_m \in \mathbb{R}^n$. The polyhedral cone or simply *cone* (resp. linear subspace) spanned by the vectors g_i will be denoted by $\langle g_1, \dots, g_m\rangle$ (resp. $(g_1, \dots, g_m)$). A cone is said to be *non-degenerate* if its generators are linearly independent (as vectors in $\mathbb{R}^n$). It is *degenerate* otherwise. A non-degenerate cone is said to be *full* (or *simplicial*) if its generators form a basis of $\mathbb{R}^n$.[1] When $C = \langle g_1, \dots, g_m\rangle = (g_1, \dots, g_m)$, C is said to be *flat*. Flatness and degeneracy should not be confused. A degenerate cone is not necessarily flat (e.g. the half line $\langle g_1, g_1\rangle$, with $g_1 \neq 0$). Recall that a cone is said to be *non-pointed* if it contains both a nonzero vector and its opposite. It is *pointed* otherwise. Thus, while a flat cone is necessarily non-pointed, the converse is not necessarily true: for instance, a (closed) half-plane is both non-pointed and non-flat. The relative interior of a cone C will be denoted by $C^\diamond$.[2] One proves that $\langle g_1, \dots, g_m\rangle^\diamond$ is the set spanned by the positive linear combination of $g_1, \dots, g_m$.

Let $\{a_1, a'_1\}, \dots, \{a_n, a'_n\}$ denote n pairs, or *dyads*, of vectors in $\mathbb{R}^n$. Let the matrices A and A' denote respectively $\begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix}$ and $\begin{pmatrix} a'_1 & \cdots & a'_n \end{pmatrix}$. Consider the mapping

$$
\begin{aligned}
[A, A'] : \quad & \{0,1\}^n \to \mathbb{R}^{n \times n} \\
& b \mapsto [A, A']_b
\end{aligned}
$$

where the ith column vector of the matrix $[A, A']_b$ is a_i if $b_i = 1$ and a'_i otherwise. A *complementary cone*, or c-cone, C is the cone spanned by the column vectors of $[A, A']_b$ for some valuation of b, that is $C = [A, A']_b(\mathbb{R}^n_+)$, the image set of $\mathbb{R}^n_+$ through the linear application represented by $[A, A']_b$. We call the column vectors of $[A, A']_b$, *the generators* of the c-cone C. A *complementary face*, or a c-face, is a face of a c-cone [Rockafellar, 1997, Section 18]. An element of a c-face is a nonnegative combination of m, $1 \le m \le n$, column vectors of $[A, A']_b$ [Rockafellar, 1997, Corollary 18.3.1]. When the subspace spanned by a face F has dimension $n - 1$, F is called a *facet*. We define similarly a *complementary linear subspace*, or c-subspace, as the set of linear combinations of m, $1 \le m \le n$, columns of $[A, A']_b$ for some $b \in \{0,1\}^n$.

Definition 1 (Covered and surrounded sets)**.** *A vector is said to be* covered *if it belongs to a c-cone. A subset of $\mathbb{R}^n$ is said to be covered if all its vectors are covered. A vector is said to be* surrounded *if it has a covered neighborhood. A subset of $\mathbb{R}^n$ is said to be surrounded if all its vectors are surrounded.*

In this work we investigate what conditions the pair $\{a_i, a'_i\}$ has to satisfy for Euclidean space to be covered. Formally, we are interested in the following problem.

Definition 2 (Q-covering)**.** *Let $\{a_i, a'_i\}$, $i = 1, \dots, n$ denote a list of n dyads of vectors in $\mathbb{R}^n$. The* Q-covering *decision problem asks whether $\mathbb{R}^n$ is covered, that is whether $\Sigma = \mathbb{R}^n$ where*

$$
\Sigma = \{a_1, a'_1\} \oplus \cdots \oplus \{a_n, a'_n\} := \bigcup_{b \in \{0,1\}^n} [A, A']_b(\mathbb{R}^n_+) \ .
$$

We adopt the sum notation for Σ in the sequel. We observe that Σ is invariant under any permutation of the indices ($\oplus$ is commutative). We think of the vector a'_i as the *dual* or the *symmetric partner* of a_i in

[1] We warn the reader that, some authors, e.g. [Cottle et al., 2009], refer to simplicial complementary cones as non-degenerate.

[2] In classical textbooks, e.g. [Rockafellar, 1997], $\langle g_1, \dots, g_m\rangle$ is denoted by $\mathrm{cone}\{g_1, \dots, g_m\}$ and the relative interior of C is denoted by $\mathrm{ri}\, C$.

the sense that Σ remains invariant when swapping a_i and a'_i for any i. The prime '$\,'\,$' can be thought of as an involutive operator providing the symmetric partner of a_i. For instance, the opposite is a very special '$\,'\,$' operator: when $a'_i = -a_i$, for all i, Σ is the partition of $\mathbb{R}^n$ into the 2^n standard orthants.

If L is a non-singular matrix, then $q \in \mathbb{R}^n$ is covered if and only if q belongs to a c-cone $\langle a_1, \ldots, a_n \rangle$ say, which is equivalent to $Lq \in \langle La_1, \ldots, La_n \rangle$. Therefore, the Q-covering problem is invariant under non-singular linear transformations of the involved vectors. If all c-cones are degenerate, $\mathbb{R}^n$, as a Baire space, cannot be covered. So for Σ to be covering, at least one c-cone must be full. There is thus no loss of generality in considering the standard basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$ respectively for $a_1, \ldots, a_n$ (equivalently A is the identity matrix) as required in the standard definition of the linear complementarity problem. Said differently, with respect to Definition 2, $\{e_1, -M_1\} \oplus \cdots \oplus \{e_n, -M_n\} = \mathbb{R}^n$ if and only if $M = (\, M_1 \cdots M_n \,)$ is a Q-matrix.

Remark 1. *Regardless of the exact geometric intersection between two c-cones, each c-cone has n (abstract)* neighbors *where two c-cones respectively generated by the column vectors of $[A, A']_b$ and $[A, A']_{b'}$ are neighbors if and only if the Hamming distance between b and b' is exactly one. Therefore, the c-cones can be put in correspondence with the vertices of an n-dimensional hypercube graph Q_n where the neighboring relationship is represented by the adjacency of the vertices in Q_n. Stitching together all c-cones along their common abstract facets, one at a time, following their neighborhood relationship, amounts to following the longest Hamiltonian cycle of Q_n (of length 2^n). In general, a family of convex sets that covers the space and for which the aforementioned neighboring relation makes sense is said to form a* Q-arrangement. *In [Kelly and Watson, 1979], Q-arrangements of full c-cones are studied.*

By introducing the equivalence relation $\simeq$ over $\mathbb{R}^n \setminus \{0\}$ defined by $u \simeq v$ if and only if $u = \lambda v$ for some positive scalar λ, one observes that if a nonzero vector q is covered then so is $q' \simeq q$. Thus one can equivalently study the covering problem of the quotient space $(\mathbb{R}^n \setminus \{0\})/\simeq$ instead of $\mathbb{R}^n$ (observing that $q = 0$ is trivially covered as it belongs to all c-cones). The quotient space $(\mathbb{R}^n \setminus \{0\})/\simeq$ is homeomorphic to the unit sphere S_{n-1} where each half-line is represented by its unit generator. Similarly, each c-cone is represented by the (possibly degenerate) spherical $(n-1)$-simplex formed by the representatives of the n generators of the cone. The spherical covering was for instance instrumental in [Kelly and Watson, 1979, Morris Jr, 1988, Cottle et al., 1981b]. (Notice, however, that the collection of the so obtained complementary simplices does not necessarily form a simplicial complex, see [Goerss and Jardine, 2012], since degeneracy and full dimensional intersections are allowed.)

In the sequel, we will find it useful to fix one or several coordinates of the Boolean vector b. We define

$$\Sigma(a_i) := \bigcup_{\substack{b \in \{0,1\}^n \\ b_i = 1}} [A, A']_b(\mathbb{R}^n_+), \qquad \Sigma(a'_i) := \bigcup_{\substack{b \in \{0,1\}^n \\ b_i = 0}} [A, A']_b(\mathbb{R}^n_+) \ .$$

Clearly, for all i, $\Sigma = \Sigma(a_i) \cup \Sigma(a'_i)$. We say that $\Sigma(a_i)$ is the set of c-cones *rooted* at a_i to make the syntactic requirement of the definition salient. (Indeed, a_i could be among the generators of a c-cone without necessarily having $b_i = 1$. For instance when $a_2 \simeq a_1$, a_1 qualifies as a generator for $\langle a'_1, a_2 \rangle$ while $b_1 = 0$.) The set of c-cones will be denoted by $\mathrm{cones}(\Sigma)$. Similarly, $\mathrm{cones}(\Sigma(a_i))$ will denote the set of c-cones rooted at a_i. We further let Σ_k, $1 \le k \le n$, denote the union of all c-faces with $m \le k$ generators: for instance Σ_1 is the set of cones $\langle a_i \rangle$, $\langle a'_i \rangle$, $i = 1, \ldots, n$. By convention, we let Σ_0 denote the set of vectors $a_1, \ldots, a_n$, $a'_1, \ldots, a'_n$.

Definition 3 (Self surrounding)**.** *The vector a_i is said to be* self surrounded *if it has a neighborhood covered by $\Sigma(a_i)$.*

Definition 4 (Lazy covering and surrounding)**.** *We say that a_i is* lazily covered *if it belongs to $\Sigma(a'_i)$. It is* lazily surrounded *if it belongs to the topological interior of a c-cone rooted at a'_i.*

We end this section by a simple definition which will be instrumental in the sequel. We use $\subseteq$ for set inclusion and $\subset$ for proper (or strict) set inclusion.

Definition 5 (Hole)**.** *A* hole *is a non-empty open connected region in $\Sigma^c := \mathbb{R}^n \setminus \Sigma$, the complement of Σ with respect to $\mathbb{R}^n$. A* maximal *hole is a hole K such that if K' is another hole, $K \subseteq K'$ implies $K' = K$.*

Since Σ is a union of finitely many closed sets, Σ^c is an open set. In general, Σ^c is a union of disconnected maximal holes. In this work, we will consider only maximal holes and refer to them simply as holes. For instance, for $n = 2$, $\Sigma = \{e_1, -e_2\} \oplus \{e_2, -e_1\}$, Σ^c has two disconnected holes, namely $\langle e_1, -e_2 \rangle^\diamond$ and $\langle -e_1, e_2 \rangle^\diamond$. A hole is not necessarily convex. For instance, when $\Sigma = \{e_1, e_1\} \oplus \{e_2, e_2\} = \langle e_1, e_2 \rangle$, $\Sigma^c = \mathbb{R}^2 \setminus \langle e_1, e_2 \rangle$ is a non-convex hole.

2 Feasibility

When $\mathbb{R}^n$ is covered by c-cones, some necessary conditions are intuitively clear and plausible. For instance, one can easily see that Euclidean space cannot be covered by Σ when the 2^n c-cones are all not full or when all vectors of Σ_0 belong to the same half-space. These conditions among others, collectively insinuate a broader necessary condition requiring the vectors of Σ_0 to be "well scattered" in the space to form enough full c-cones that are in turn sufficiently distributed to achieve a covering. For instance, it is well known that at least $n+1$ full cones are required to cover $\mathbb{R}^n$: a full cone C with an additional vector in the topological interior of $-C$ partition the space in $n+1$ full cones. [3] We shall see that, in the context of this paper, a similar *separation* property follows from Proposition 4. Let $\Gamma = \langle a_1, a'_1, \ldots, a_n, a'_n \rangle$. As we are working in dimension n, Γ can be seen as the union of $\binom{2n}{n}$ cones generated by any n vectors in Σ_0. The set of such cones will be denoted by cones(Γ). We proceed to investigate under which conditions Γ covers $\mathbb{R}^n$ ensuring feasibility. We do this in Proposition 1 after stating a technical lemma (akin to fan and barycentric triangulation) characterizing flat cones as those having 0 in their relative interior.

Lemma 1. *Suppose that $0 \in \langle g_1, \ldots, g_{m+1} \rangle^\diamond$, $1 \le m \le n$, then $\langle g_1, \ldots, g_{m+1} \rangle$ is flat of dimension at most m. Moreover, $\langle g_1, \ldots, g_{m+1} \rangle = \cup_i G_i$ where G_i, $1 \le i \le m+1$, is the cone generated by $\{g_1, \ldots, g_{m+1}\} \setminus \{g_i\}$.*

Proof. There exist $\alpha_1, \ldots, \alpha_{m+1} > 0$ such that $0 = \sum_{i=1}^{m+1} \alpha_i g_i$. Thus $(g_1, \ldots, g_{m+1}) = (g_1, \ldots, g_m)$. Let $x \in (g_1, \ldots, g_m)$. Then $x = \sum_{i=1}^{m} \lambda_i g_i$ where $\lambda_i \in \mathbb{R}$. If $\lambda_i \ge 0$ for all $1 \le i \le m$, then $x \in \langle g_1, \ldots, g_m \rangle \subseteq \langle g_1, \ldots, g_{m+1} \rangle$. Otherwise, there exists a non-empty set of indices J such that $\lambda_j < 0$ for all $j \in J$. Let $\lambda = \max_{j \in J}\{\frac{-\lambda_j}{\alpha_j}\} > 0$. So, for all i, $\lambda_i + \lambda\alpha_i \ge 0$. Then

$$x = x + \lambda \times 0 = \sum_{i=1}^{m} \lambda_i g_i + \lambda \sum_{i=1}^{m+1} \alpha_i g_i = \lambda\alpha_{m+1} g_{m+1} + \sum_{i=1}^{m} (\lambda_i + \lambda\alpha_i) g_i$$

and $x \in \langle g_1, \ldots, g_{m+1} \rangle$. Therefore $(g_1, \ldots, g_{m+1}) = (g_1, \ldots, g_m) \subseteq \langle g_1, \ldots, g_{m+1} \rangle \subseteq (g_1, \ldots, g_{m+1})$. This proves the first part of the lemma. For the second part, if J is empty, then $x \in G_{m+1}$, otherwise, if we let $k \in J$ denotes the index for which $\lambda = \frac{-\lambda_k}{\alpha_k}$, then one sees that $x \in G_k$. Thus $\langle g_1, \ldots, g_{m+1} \rangle \subseteq G_{m+1} \cup (\cup_{j \in J} G_j) \subseteq \cup_{i=1}^{m+1} G_i$. The converse inclusion is immediate and the equality holds as stated. ☐

Proposition 1. *Let $k \ge n+1$ and $g_1, \ldots, g_k$ denote k nonzero vectors of $\mathbb{R}^n$. If $\mathbb{R}^n = \langle g_1, \ldots, g_k \rangle$ then there exist $i_1, \ldots, i_{m+1}$ with $1 \le m \le n$ such that $\langle g_{i_1}, \ldots, g_{i_{m+1}} \rangle$ is a flat of dimension m.*

Proof. The proof is by induction on n. For $n = 1$, suppose $\mathbb{R} = \langle g_1, \ldots, g_k \rangle$ with $k \ge 2$. Then $0 \in \langle g_1, \ldots, g_k \rangle^\diamond$ and there exist $\lambda_1, \ldots, \lambda_k > 0$ such that $\sum_i \lambda_i g_i = 0$. If $g_i < 0$ for all i we get a contradiction, so there exists $g_j > 0$. Similarly, if $g_i > 0$ for all i we also get a contradiction, so there exists $g_\ell < 0$ and we have $\mathbb{R} = \langle g_j, g_\ell \rangle$ and $m = n = 1$. Let π denote the orthogonal projection onto the hyperplane $g_1^\perp$ ($g_1 \neq 0$ by hypothesis). Let $x \in g_1^\perp$ which is a subset of $\mathbb{R}^n = \langle g_1, \ldots, g_k \rangle$. Then $x = \sum_{i=1}^{k} \lambda_i g_i$ with $\lambda_i \ge 0$. Thus $x = \pi(x) = \sum_{i=2}^{k} \lambda_i \pi(g_i)$ and therefore $g_1^\perp \subseteq \langle \pi(g_2), \ldots, \pi(g_k) \rangle$. The converse inclusion is immediate by definition of π. Thus $g_1^\perp = \langle \pi(g_2), \ldots, \pi(g_k) \rangle$. By the induction hypothesis, there exists $1 \le m \le n-1$ such that $\langle \pi(g_{i_2}), \ldots, \pi(g_{i_{m+2}}) \rangle$ is a flat of dimension m. In particular $\pi(g_{i_2}), \ldots, \pi(g_{i_{m+1}})$ are linearly independent. Assume without loss of generality that $i_j = j$. We have $0 = \sum_{j=2}^{m+2} \alpha_j \pi(g_j)$, with $\alpha_j > 0$. Moreover, for each g_i, there exists $\theta_i \in \mathbb{R}$ such that $g_i = \theta_i g_1 + \pi(g_i)$. Thus $0 = \sum_{j=2}^{m+2} \alpha_j (g_j - \theta_j g_1) = (-\sum_{j=2}^{m+2} \alpha_j \theta_j) g_1 + \sum_{j=2}^{m+2} \alpha_j g_j$. Let $\gamma = -\sum_{j=2}^{m+2} \alpha_j \theta_j$. If $\gamma > 0$, then by Lemma 1, $(g_1, \ldots, g_{m+2}) = (g_1, \ldots, g_{m+1}) = \langle g_1, \ldots, g_{m+2} \rangle$. We further prove that $g_1, \ldots, g_{m+1}$ are linearly independent. Suppose $0 = \sum_{i=1}^{m+1} \sigma_i g_i$ then $0 = \sum_{i=2}^{m+1} \sigma_i \pi(g_i)$ and since $\pi(g_2), \ldots, \pi(g_{m+1})$ are linearly independent, $\sigma_i = 0$ for all $2 \le i \le m+1$. Thus $0 = \sigma_1 g_1$, $\sigma_1 = 0$, and $\langle g_1, \ldots, g_{m+2} \rangle$ is a flat of dimension $m+1$, which is what we wanted to prove. If $\gamma = 0$, then by Lemma 1, $(g_2, \ldots, g_{m+1}) = (g_2, \ldots, g_{m+2}) = \langle g_2, \ldots, g_{m+2} \rangle$. Thus $\langle g_2, \ldots, g_{m+2} \rangle$ is a flat of dimension m. If $\gamma < 0$, then $g_1 \in \langle g_2, \ldots, g_{m+2} \rangle$ and $\mathbb{R}^n = \langle g_2, \ldots, g_k \rangle$. We repeat the same reasoning with g_2 and either we find a flat of dimension m or $m+1$ or g_2 can be also removed from the list of generators till eventually reaching $n+1$ generators for $\mathbb{R}^n$ at which point $m = n$ and we are done. ☐

Proposition 1 will be used in Section 5 to characterize the feasibility in dimension 3. In the sequel, assuming $\Gamma = \mathbb{R}^n$, we show in Proposition 2 that the space can be covered by what we term *minimal cones*. Lemma 2 is akin to the *anti-exchange* property of $\mathbb{R}^n$ [Coppel, 1998, Chapter I, §3] and Lemma 3 can be seen as an application of the conical version of Caratheodory's theorem.

Lemma 2. *Let $u, v, g_1, \ldots, g_m \in \mathbb{R}^n$, $1 \le m \le n-1$, such that $u, g_1, \ldots, g_m$ or $v, g_1, \ldots, g_m$ are linearly independent. Then $u \in \langle v, g_1, \ldots, g_m \rangle$ and $v \in \langle u, g_1, \ldots, g_m \rangle$ if and only if $u \simeq v$. When $m = 1$, the linear independence condition can be dropped.*

Proof. There exist $\alpha, \lambda, \alpha_i, \lambda_i \ge 0$ such that $u = \alpha v + \sum_{i=1}^{m} \alpha_i g_i$ and $v = \lambda u + \sum_{i=1}^{m} \lambda_i g_i$. Assume $v, g_1, \ldots, g_m$ are linearly independent (otherwise, swap u and v in what follows). We get $(1 - \lambda\alpha)v = \sum_{i=1}^{m} (\lambda\alpha_i + \lambda_i) g_i$. The linear independence implies $1 - \alpha\lambda = 0$ and $\alpha\lambda_i + \alpha_i = 0$ for all i. Thus $\alpha, \lambda > 0$ and $\alpha_i = \lambda_i = 0$ for all i proving the statement. For $m = 1$, dropping the linear independence hypothesis, the proof is by case distinction. If $\lambda\alpha_1 + \lambda_1 = 0$ then $\alpha, \lambda > 0$ and $\alpha_1 = \lambda_1 = 0$ and therefore $v \simeq u$. Likewise, if $\alpha\lambda_1 + \alpha_1 = 0$, $u \simeq v$. Suppose $\lambda\alpha_1 + \lambda_1 > 0$ and $\alpha\lambda_1 + \alpha_1 > 0$. If $1 - \lambda\alpha < 0$ then $u \simeq -g_1$ and $v \simeq -g_1$ so $u \simeq v$. Otherwise $u \simeq v \simeq g_1$. ☐

[3] This fact can be seen as a corollary of Stiemke's theorem [Cottle et al., 2009, Theorem 2.7.12].

Lemma 3. *Let $G = \langle g_1, \ldots, g_n \rangle$ denote a cone in $\mathbb{R}^n$. Let g be a nonzero vector in G and let G_i denote the cone generated by $\{g, g_1, \ldots, g_n\} \setminus \{g_i\}$, $1 \le i \le n$. Then $G = \cup_i G_i$. If moreover G is full then there exists a non-empty set of indices $J \subseteq \{1, \ldots, n\}$ such that $G = \cup_{j \in J} G_j$, G_j is full, and $G_j \subseteq G$, for all $j \in J$. Moreover $G_j = G$ if and only if $g \simeq g_j$.*

Proof. We have $g = \sum_{i=1}^n \alpha_i g_i$ for some nonnegative coefficients α_i and, since $g \neq 0$, there exists a non-empty set of indices $J \subseteq \{1, \ldots, n\}$ such that $\alpha_j > 0$ for all $j \in J$. Let $x \in G$, i.e. $x = \sum_{i=1}^n \beta_i g_i$ for some nonnegative coefficients β_i. Let $\lambda = \min_{j \in J} \{ \frac{\beta_j}{\alpha_j} \} = \frac{\beta_k}{\alpha_k}$ for some $k \in J$. We then have, for all $i = 1, \ldots, n$, $\beta_i - \lambda \alpha_i \ge 0$ and

$$x = \sum_{i=1}^n \beta_i g_i = \lambda g + \sum_{i=1}^n (\beta_i - \lambda \alpha_i) g_i = \lambda g + \sum_{i \neq k} (\beta_i - \lambda \alpha_i) g_i \ .$$

Thus $x \in G_k$ and $G \subseteq \cup_{j \in J} G_j \subseteq \cup_i G_i$. This proves the first statement. Suppose G is full and G_j is degenerate for an index $j \in J$, then g belongs to the hyperplane H_j generated by g_i, $i \neq j$. [4] But then $g_j \in H_j$ (because $\alpha_j > 0$) and G is itself degenerate, a contradiction. Thus G_j is full for all $j \in J$. Since $g \in G$, $G_j \subseteq G$ for all j. If in addition $G \subseteq G_j$ for some index j, then by Lemma 2 (applied to g and g_j as u and v), this is equivalent to $g \simeq g_j$. ☐

Proposition 2. *Assume $\Gamma = \mathbb{R}^n$. Then for any $x \in \mathbb{R}^n$, there exists a* minimal cone *$G \in \mathrm{cones}(\Gamma)$ containing x, that is G is full and for any other full cone $G' \in \mathrm{cones}(\Gamma)$, $G' \subseteq G$ implies $G' = G$.*

Proof. Since $\Gamma = \mathbb{R}^n$, by the Caratheodory theorem, there exists a full cone G_1 with n generators in Σ_0 such that $x \in G_1$ (G_1 needs not be unique). Let $g_1, \ldots, g_n \in \Sigma_0$ denote the n generators of G_1 and suppose there exists a nonzero vector $g_{n+1} \in \Sigma_0 \setminus \{g_1, \ldots, g_n\}$ such that $g_{n+1} \in G_1$ and $g_{n+1} \not\simeq g_i$ for all $1 \le i \le n$. By Lemma 3, there exists a full cone $G_2 \subset G_1$ containing x. Suppose without loss of generality that $g_1 \notin G_2$ (such a vector must exist by construction of G_2), so $G_2 = \langle g_2, \ldots, g_{n+1} \rangle$. If G_2 itself contains a vector $g_{n+2} \in \Sigma_0 \setminus \{g_1, \ldots, g_{n+1}\}$ such that $g_{n+2} \not\simeq g_i$ for all $2 \le i \le n+1$, then again by Lemma 3, there exists a full cone $G_3 \subset G_2$ containing x. At step k, we ask if G_k contains a vector $g_{n+k} \in \Sigma_0 \setminus \{g_1, \ldots, g_{n+k-1}\}$ such that $g_{n+k} \not\simeq g_i$ for all $k \le i \le n+k-1$. The process must terminate after at most n steps since $\Sigma_0 \setminus \{g_1, \ldots, g_{n+n}\}$ is empty. We let $G = G_k$, $1 \le k \le n$, where G_k does not contain any vector from Σ_0 (apart from those in the equivalence classes of its generators with respect to $\simeq$). In particular there is no other cone $G' \in \mathrm{cones}(\Gamma)$ such that $G' \subset G$. ☐

Minimal cones play an important role in locating holes. They will be for instance instrumental in Section 4 to characterize Q-covering for $n = 3$.

Proposition 3. *Assume $\Gamma = \mathbb{R}^n$. If $\Sigma \subset \mathbb{R}^n$ then there exists a minimal cone $G \in \mathrm{cones}(\Gamma) \setminus \mathrm{cones}(\Sigma)$ such that $G \cap \Sigma^c$ is non-empty.*

Proof. Let $x \in \Sigma^c$. By Proposition 2, there exists a minimal cone $G \in \mathrm{cones}(\Gamma)$ containing x. The cone G cannot be a c-cone since it contains x. So $G \notin \mathrm{cones}(\Sigma)$. Clearly we have $x \in G \cap \Sigma^c$ and the intersection is non-empty. ☐

Remark 2. *One can tighten the statement of Proposition 3 by saying that $G^\diamond \cap \Sigma^c$ is a non-empty open set. If x belongs to the interior of G then $G^\diamond \cap \Sigma^c$ is non-empty. Otherwise, since Σ^c is an open set, a small perturbation of x remains in Σ^c while avoiding the boundaries of G. Thus $G^\diamond \cap \Sigma^c$ is also non-empty.*

While Γ and Σ are both finite unions of closed convex cones, they are only seemingly similar. An important difference between the two being convexity: Γ is convex by definition, but Σ is not. This difference introduces a substantial complexity to the covering problem. For instance, the extreme rays of Γ (if any) are necessarily among the generators of Γ [Rockafellar, 1997, Corollary 18.3.1]. This same property is far from obvious for Σ as we shall see in Section 4. The next section investigates some interesting properties of Σ.

3 Solvability

We start by proving a necessary condition for $\mathbb{R}^n$ to be covered, namely, that for all i, $a_i, a'_i \neq 0$ and $a'_i \not\simeq a_i$ (geometrically, this means that $\langle a_i, a'_i \rangle$ is not reduced to the origin nor is it a half-line).

Lemma 4. *Assume there exists at least one full c-cone. If all full c-cones meet at a nonzero vector then $\Sigma \subset \mathbb{R}^n$. Moreover, if there exists an index i such that $a_i = 0$ or $a'_i \simeq a_i$ then $\Sigma \subset \mathbb{R}^n$.*

Proof. If $a_i = a'_i = 0$ for some index i, then all c-cones are degenerate contradicting the assumption. Assume next that, there exists at least one full c-cone and that for all i, either $a_i \neq 0$ or $a'_i \neq 0$. Suppose there exists a vector $p \neq 0$ that belongs to all full c-cones. The full c-cones cannot cover $-p$ as this would contradict the non-degeneracy assumption. This means that an open neighborhood U of $-p$ is left to be covered by a finite number of degenerate cones, which is impossible ($\mathbb{R}^n$ is a Baire space). [5] In particular, if $a_i = 0$ and $a'_i \neq 0$,

[4] In this paper, a *hyperplane* denotes a linear (or vector) subspace of dimension $n - 1$.

[5] Every complete metric space (such as Euclidean space) is a Baire space in which countable unions of closed sets with empty interior have also an empty interior.

then all full c-cones meet at $p \simeq a'_i$ and the covering is impossible. Moreover, when $a_i \neq 0$ and $a'_i \simeq a_i$ then all c-cones meet at $p \simeq a_i$ and the covering is also impossible. $\square$

Proposition 4 below states a necessary condition on pairs $\{a_i, a'_i\}$ drawing upon an existing result by [Samelson et al., 1958] characterizing the partition of Euclidean space into c-cones using *separation* which we now define.

Definition 6 (Separation)**.** *Let H denote a hyperplane and let h denote a nonzero vector orthogonal to H. Two vectors u, v are said to be* separated *by H if and only if the scalar products $u.h$ and $v.h$ are not zero and have opposite signs. (Geometrically, u and v belong each to a distinct open half-space bounded by H.)*

3.1 A Necessary Separation Condition

Proposition 4. *If $\Sigma = \mathbb{R}^n$ then for each index $i = 1, \ldots, n$ there must exist a complementary hyperplane (that is a c-subspace of dimension $n-1$) separating the pair $\{a_i, a'_i\}$.*

Proof. By Lemma 4, for all i, $a_i, a'_i \neq 0$ and $a_i \not\simeq a'_i$. The proof is by contradiction. Fix i and let V_r, $r = 1, \ldots, 2^{n-1}$, denote the c-subspaces generated by the remaining $(n-1)$ vectors a_j, a'_j, $j \neq i$. If $\dim(V_r) < n-1$ for all r, then all c-cones are degenerate and $\Sigma \subset \mathbb{R}^n$. Thus, there must exist a non-empty subset of indices $S \subseteq \{1, \ldots, 2^{n-1}\}$ such that V_s is a hyperplane for all $s \in S$. Assume that the pair $\{a_i, a'_i\}$ is not separated by any hyperplane V_s, $s \in S$. If $a_i, a'_i \in V_s$ for all s, then again, all c-cones are degenerate and $\Sigma \subset \mathbb{R}^n$. Thus, there must exist a set of indices $K \subseteq S$ such that $a_i \notin V_k$ or $a'_i \notin V_k$ for all $k \in K$. Let $V_k(i)$, $k \in K$, denote the closed half-space bounded by V_k and containing either a_i or a'_i in its interior. We thus have $a_i, a'_i \in V := \cap_{k \in K} V_k(i)$. The boundary of V, denoted ∂V, is a subset of the union of the boundaries of $V_k(i)$. Since V is a (closed) convex cone then $\langle a_i, a'_i \rangle \subseteq V$. We show by contradiction that the topological interior of V is non-empty. Assume it is empty. Then V is equal to its boundary ∂V and $V = \partial V \subseteq \cup_{k \in K} V_k$. Since $a'_i \in V$, there exists an index $k_1 \in K$ such that $a'_i \in V_{k_1}$ and $a_i \notin V_{k_1}$. Thus $\dim(a_i, a'_i) = 2$ and we construct a sequence of vectors $p_j \neq 0$, $1 \leq j \leq |K|$, such that $p_j \in \langle a_i, p_{j-1} \rangle^\diamond$ with $p_1 = a'_i$. For each j, $p_j \in V$, so there exists an index $k_j \in K$ such that $p_j \in V_{k_j}$. If $k_j = k_\ell$ for $1 \leq j, \ell \leq |K|$, $j \neq \ell$, then $p_j, p_\ell \in V_{k_j} = V_{k_\ell}$, and $(a_i, a'_i) = (p_j, p_\ell) \subseteq V_{k_j}$. But then $a_i, a'_i \in V_{k_j}$, contradicting the fact that $k_j \in K$. Thus $k_1, \ldots, k_{|K|}$ are all distinct making $K = \{k_1, \ldots, k_{|K|}\}$. Now consider a vector $p \neq 0$ in $\langle a_i, p_{|K|} \rangle^\diamond$. Then there exists $k_j \in K$ such that $p \in V_{k_j}$ and therefore $(a_i, a'_i) = (p, p_j) \subseteq V_{k_j}$, a contradiction. So the topological interior of V is non-empty. We denote it by $V^\diamond$. We show next that $-V^\diamond$ is a hole. First, by definition of $V^\diamond$, $V^\diamond \subseteq V_k(i)$, thus $-V^\diamond \subseteq V_k(i)^c$, for each $k \in K$. Therefore $-V^\diamond \subseteq \cap_k V_k(i)^c = (\cup_k V_k(i))^c$. Second, let $C(a_i, V_r)$ denote the c-cone spanned by a_i and the generators of V_r. For $k \in K$, since V_k does not separate the pair $\{a_i, a'_i\}$ then $C(a_i, V_k) \cup C(a'_i, V_k) \subseteq V_k(i)$. Thus

$$\Sigma_K := \cup_{k \in K} \big(C(a_i, V_k) \cup C(a'_i, V_k) \big) \subseteq \cup_{k \in K} V_k(i),$$

and $\big(\cup_{k \in K} V_k(i)\big)^c \subseteq (\Sigma_K)^c$. Thus $-V^\diamond \subseteq (\Sigma_K)^c$. Finally, all is left to cover $-V^\diamond$ is a finite set of degenerate c-cones, namely $C(a_i, V_r)$ and $C(a'_i, V_r)$, $r \notin K$. Covering an open set with a finite number of degenerate cones is impossible (by Baire's theorem). So there must exist an index $s \in S$ such that V_s is a hyperplane separating the pair $\{a_i, a'_i\}$. $\square$

With respect to the notations of the proof of Proposition 4, the seminal work of [Samelson et al., 1958] shows that Σ partitions $\mathbb{R}^n$ if and only if, for each index $i = 1, \ldots, n$ and for each $r = 1, \ldots, 2^{n-1}$, the c-subspace V_r is a hyperplane separating the pair $\{a_i, a'_i\}$. Proposition 4 shows that, for Σ to be (only) covering, it is necessary that for each index i at least one c-subspace V_r is a complementary hyperplane separating the pair $\{a_i, a'_i\}$. We know that this necessary condition is not sufficient in general to prove that Σ is covering. For instance, in the plane, let $a'_1 = -e_1 + e_2$ and $a'_2 = -e_1$. Then $\Sigma = \{e_1, a'_1\} \oplus \{e_2, a'_2\}$ covers only the closed upper half-plane. The pair $\{e_1, a'_1\}$ is separated by the line (e_2) and the pair $\{e_2, a'_2\}$ is separated by the line (a'_1).

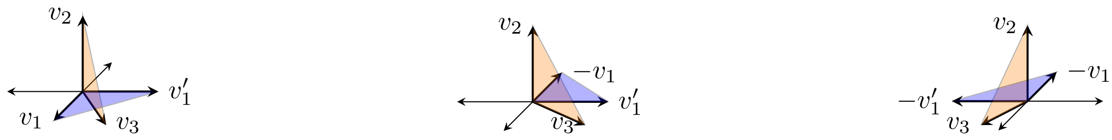
In the next section we shift focus on how c-facets (instead of c-hyperplanes) might "separate" pairs via intersections.

3.2 Dyadic Covering

We shall see several interesting properties concerned with intersections of c-facets with the cone $\langle a_i, a'_i \rangle$. We start by stating three generic relevant intersections (all depicted in Fig. 1) that will be instrumental in the rest of the paper. Lemmas 5 and 6 play a role in the forthcoming Propositions 5 and 6. Lemma 7 shall be used in Section 4.

Lemma 5 (Front Intersection (cf. Fig 1a))**.** *Fix $m \geq 2$ and let $C = \langle v_1, v_2, \ldots, v_m \rangle$ and $C' = \langle v'_1, v_2, \ldots, v_m \rangle$. If there exists a nonzero vector in $\langle v'_1, v_1 \rangle \cap \langle v_2, \ldots, v_m \rangle$ then $\langle v'_1, v_1, v_2, \ldots, v_m \rangle = C \cup C'$.*

Proof. Let $c \in \langle v'_1, v_1, v_2, \ldots, v_m \rangle$. There exist $\alpha'_1, \alpha_1, \ldots, \alpha_m \geq 0$ such that $c = \alpha_1 v_1 + \alpha'_1 v'_1 + \sum_{i \geq 2} \alpha_i v_i$. By assumption, there exist $\lambda'_1, \lambda_1, \ldots, \lambda_m \geq 0$, $(\lambda_1, \lambda'_1) \neq (0, 0)$, such that $\lambda_1 v_1 + \lambda'_1 v'_1 = \sum_{i \geq 2} \lambda_i v_i$. Suppose



(a) Front (b) Side (c) Back

Figure 1: Conical intersections

$\lambda_1, \lambda_1' > 0$ and let $\beta = \min\{\frac{\alpha_1}{\lambda_1}, \frac{\alpha_1'}{\lambda_1'}\} \geq 0$. Then

$$
\begin{aligned}
c = c + \beta \times 0 &= \alpha_1 v_1 + \alpha_1' v_1' + \sum_{i\geq 2} \alpha_i v_i + \beta(-\lambda_1 v_1 - \lambda_1' v_1' + \sum_{i\geq 2} \lambda_i v_i) \\
&= (\alpha_1 - \beta\lambda_1) v_1 + (\alpha_1' - \beta\lambda_1') v_1' + \sum_{i\geq 2} (\alpha_i + \beta\lambda_i) v_i \ .
\end{aligned}
$$

By definition of β, the coefficients of v_1 and v_1' are both nonnegative and one of them has to vanish. Thus $c \in C \cup C'$. The same holds when $\lambda_1 = 0$ (resp. $\lambda_1' = 0$) by taking β as $\frac{\alpha_1'}{\lambda_1'}$ (resp. $\frac{\alpha_1}{\lambda_1}$). The converse inclusion is immediate. □

Lemma 6 (Side Intersection (cf. Fig 1b))**.** *Fix $m \geq 2$ and let $C = \langle v_1, v_2, \ldots, v_m\rangle$ and $C' = \langle v_1', v_2, \ldots, v_m\rangle$. If $\langle -v_1, v_1'\rangle^\diamond \cap \langle v_2, \ldots, v_m\rangle$ is non-empty then $\langle v_1, v_1', v_2, \ldots, v_m\rangle \subseteq C$.*

Proof. By assumption, there exist $\alpha_1', \alpha_1, \ldots, \alpha_m \geq 0$ such that $\alpha_1(-v_1) + \alpha_1' v_1' = \sum_{i\geq 2} \alpha_i v_i$ with $\alpha_1, \alpha_1' > 0$. Thus $v_1' \in C$ and $\langle v_1, v_1', v_2, \ldots, v_m\rangle \subseteq C$. □

Lemma 7 (Back Intersection (cf. Fig 1c))**.** *Let $G = \langle v_1, v_1', v_2, \ldots, v_{n-1}\rangle$ denote a full cone in $\mathbb{R}^n$. Suppose there exists a vector $p \neq 0$ such that $p \in \langle -v_1, -v_1'\rangle \cap \langle v_2, \ldots, v_n\rangle$ then $G^\diamond \subseteq (\langle v_1, v_2, \ldots, v_n\rangle \cup \langle v_1', v_2, \ldots, v_n\rangle)^c$.*

Proof. There exist $\lambda_1, \lambda_1', \lambda_2, \ldots, \lambda_n \geq 0$, $(\lambda_1, \lambda_1') \neq (0, 0)$ such that

$$
p = \lambda_1(-v_1) + \lambda_1'(-v_1') = \sum_{i=2}^{n} \lambda_i v_i \ . \tag{1}
$$

If $\lambda_n = 0$ then G is degenerate. Thus $\lambda_n > 0$. Let $g \in G^\diamond$. Then $g = \alpha_1 v_1 + \alpha_1' v_1' + \sum_{i=2}^{n-1} \alpha_i v_i$ with $\alpha_1, \alpha_1', \alpha_2, \ldots, \alpha_{n-1} > 0$. Suppose that $g \in \langle v_1, \ldots, v_n\rangle$, then there exist $\beta_i \geq 0$ such that $g = \sum_{i=1}^{n} \beta_i v_i$. Thus $\alpha_1 v_1 + \alpha_1' v_1' + \sum_{i=2}^{n-1} \alpha_i v_i = \sum_{i=1}^{n} \beta_i v_i$ and

$$
\alpha_1' v_1' + \sum_{i=1}^{n-1} (\alpha_i - \beta_i) v_i = \beta_n v_n \ . \tag{2}
$$

If $\beta_n = 0$ then $\alpha_1' = 0$ (because $v_1, v_1', v_2, \ldots, v_{n-1}$ are linearly independent), which contradicts $\alpha_1' > 0$. Thus $\beta_n > 0$. Since $\lambda_n, \beta_n > 0$ we can eliminate v_n in Eq. 1 and Eq. 2. We get

$$
(\beta_n \lambda_1' + \lambda_n \alpha_1')(-v_1') + \sum_{i=1}^{n-1} (\beta_n \lambda_i + \lambda_n(\alpha_i - \beta_i))(-v_i) = 0 \ .
$$

As $v_1, v_1', v_2, \ldots, v_{n-1}$ are linearly independent, all the coefficients must vanish. In particular, $\beta_n \lambda_1' + \lambda_n \alpha_1' = 0$ which implies $\lambda_n \alpha_1' = 0$, a contradiction. Thus $g \notin \langle v_1, \ldots, v_n\rangle$. Swapping v_1 and v_1' in the discussion above, we prove that $g \notin \langle v_1', v_2, \ldots, v_n\rangle$ and therefore $G^\diamond \subseteq (\langle v_1, \ldots, v_n\rangle \cup \langle v_1', v_2, \ldots, v_n\rangle)^c$ as stated. □

Proposition 5 states an interesting covering property of the cone $\langle a_i, a_i'\rangle$. Its proof requires the following technical lemma. [6]

Lemma 8. *Let $G = \langle g_1, \ldots, g_m\rangle$, $m \geq 2$, denote a cone in $\mathbb{R}^n$ and let $q \notin G$ be such that $\langle g_1, q\rangle$ is non-flat. Suppose there exists a nonzero vector in $\langle g_1, q\rangle^\diamond \cap G$ then there exists a nonzero vector in $\langle g_2, \ldots, g_m\rangle \cap \langle g_1, q\rangle^\diamond$.*

Proof. Let $p' \neq 0$ be in $\langle g_1, q\rangle^\diamond \cap G$. Then there exists $\lambda_1, \lambda > 0$ and $\alpha_1, \ldots, \alpha_m \geq 0$ such that $p' = \lambda_1 g_1 + \lambda q = \sum_{i=1}^{m} \alpha_i g_i$. Since $q \notin G$ then $0 < \lambda_1 - \alpha_1$. Let $p = (\lambda_1 - \alpha_1) g_1 + \lambda q$. Then, $p \in \langle g_2, \ldots, g_m\rangle \cap \langle g_1, q\rangle^\diamond$. If $p = 0$ then $-g_1 \simeq q$ ($q \neq 0$ because $q \notin G$), making $\langle g_1, q\rangle$ flat. Thus $p \neq 0$. □

Proposition 5. *The cone $\langle a_i, a_i'\rangle$ cannot be partially covered, i.e. either $\langle a_i, a_i'\rangle \subseteq \Sigma$ or $\langle a_i, a_i'\rangle^\diamond \subseteq \Sigma^c$.*

[6] The original proof was by induction on m. The provided shorter constructive proof was suggested by Jean-Charles Gilbert in a private communication with the first author.

Proof. We fix i to 1 for clarity. If $\dim(a_1, a_1') \leq 1$, then $\langle a_1, a_1'\rangle \subseteq \Sigma$. Assume next that $\dim(a_1, a_1') = 2$ (in particular $a_1' \not\simeq -a_1$). Suppose there exists $a \in \langle a_1, a_1'\rangle^\diamond$ such that $a \in \Sigma$. So a belongs to a c-cone $C = \langle a_1, \ldots, a_n\rangle$ say (otherwise swap a_1 and a_1' in what follows). If $a_1' \in C$ then, by convexity of C, $\langle a_1, a_1'\rangle \subseteq C \subseteq \Sigma$. Assume $a_1' \notin C$. By Lemma 8, $\langle a_1, a_1'\rangle^\diamond$ intersects the face $\langle a_2, \ldots, a_n\rangle$ of C in a nonzero vector. By Lemma 5, $\langle a_1', a_1, \ldots, a_n\rangle \subseteq \langle a_1, a_2, \ldots, a_n\rangle \cup \langle a_1', a_2, \ldots, a_n\rangle$ which implies $\langle a_1, a_1'\rangle \subseteq \Sigma$. Finally, if such an a does not exist then $\langle a_1, a_1'\rangle^\diamond \subseteq \Sigma^c$, which ends the proof. ☐

We further characterize when the cone $\langle a_i, a_i'\rangle$ is covered. It turns out that such a covering requires specific intersections with c-faces.

Proposition 6. *Let $a_i, a_i' \in \mathbb{R}^n$. Then $\langle a_i, a_i'\rangle \subseteq \Sigma$ if and only if either $\dim(a_i, a_i') \leq 1$ or $\dim(a_i, a_i') = 2$ and there exists a c-face $F_i := \langle a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n\rangle$ and a vector $p \neq 0$ such that one of the following conditions occurs:*

$$1.\ p \in \langle a_i, a_i'\rangle \cap F_i, \quad 2.\ p \in \langle -a_i, a_i'\rangle^\diamond \cap F_i, \quad 3.\ p \in \langle a_i, -a_i'\rangle^\diamond \cap F_i\ .$$

In all cases, one has $\langle a_i', a_1, \ldots, a_n\rangle \subseteq \Sigma$.

Proof. We fix i to 1 for clarity. Suppose $\langle a_1, a_1'\rangle \subseteq \Sigma$ and that $\dim(a_1, a_1') = 2$. Let $a \in \langle a_1, a_1'\rangle^\diamond$, that is $a = \alpha_1 a_1 + \alpha_1' a_1'$, $\alpha_1, \alpha_1' > 0$. Since $a \in \Sigma$, then either (i) $a \in \langle a_1, \ldots, a_n\rangle$ or (ii) $a \in \langle a_1', \ldots, a_n\rangle$ for some c-face $F_1 = \langle a_2, \ldots, a_n\rangle$. For (i), one gets $\alpha_1 a_1 + \alpha_1' a_1' = \sum_{i=1}^n \beta_i a_i$ with $\beta_1, \ldots, \beta_n \geq 0$. Thus $(\alpha_1 - \beta_1)a_1 + \alpha_1' a_1' = \sum_{i=2}^n \beta_i a_i$. Let $p = (\alpha_1 - \beta_1)a_1 + \alpha_1' a_1'$. Then $p \neq 0$ as otherwise $\dim(a_1, a_1') \leq 1$.

- If $\alpha_1 - \beta_1 \geq 0$ then $p \in \langle a_1, a_1'\rangle \cap F_1$ (case 1 of the statement).
- If $\alpha_1 - \beta_1 < 0$ then $p \in \langle -a_1, a_1'\rangle^\diamond \cap F_1$ (case 2 of the statement).

The same discussion holds by swapping a_1 and a_1' for (ii), leading to case 3. For the converse, if $\dim(a_1, a_1') \leq 1$ then it is immediate that $\langle a_1, a_1'\rangle \subseteq \Sigma$. Otherwise, one of the stated conditions holds. If the first condition holds, then by Lemma 5 $\langle a_1, a_1', a_2, \ldots, a_n\rangle \subseteq \langle a_1, a_2, \ldots, a_n\rangle \cup \langle a_1', a_2, \ldots, a_n\rangle \subseteq \Sigma$. If the second condition holds, then by Lemma 6, $\langle a_1, a_1', a_2, \ldots, a_n\rangle \subseteq \langle a_1, \ldots, a_n\rangle \subseteq \Sigma$. If the third condition holds, then by Lemma 6 $\langle a_1, a_1', a_2, \ldots, a_n\rangle \subseteq \langle a_1', \ldots, a_n\rangle \subseteq \Sigma$ and $\langle a_1, a_1'\rangle \subseteq \Sigma$, as stated. ☐

3.3 Surrounding

When the cone $\langle a_i, a_i'\rangle$ is degenerate, its front, side and back intersections with c-faces become ill-defined. We argue that these intersections are only relevant whenever $\langle a_i, a_i'\rangle$ is non-degenerate. We have already seen in Lemma 4 that $a_i \not\simeq a_i'$ for all i is a necessary condition for Σ to be covering. We shall see next that when $\langle a_i, a_i'\rangle$ is flat, we can drop the pair altogether and reduce the dimension of the Q-covering problem by 1.

Proposition 7. *Assume that $a_i \neq 0$ and suppose that $\langle a_i, a_i'\rangle$ is flat. Let $a_i^\perp$ denote the hyperplane orthogonal to a_i, and let $\bar{v}$ denote the orthogonal projection of a vector v onto $a_i^\perp$. Then $\Sigma = \mathbb{R}^n$ if and only if the sum $\{\bar{a}_1, \bar{a}_1'\} \oplus \cdots \oplus \{\bar{a}_{i-1}, \bar{a}_{i-1}'\} \oplus \{\bar{a}_{i+1}, \bar{a}_{i+1}'\} \oplus \cdots \oplus \{\bar{a}_n, \bar{a}_n'\}$ covers $a_i^\perp$ (which is isomorphic to $\mathbb{R}^{n-1}$).*

Proof. For clarity we fix i to 1. Let $h \in a_1^\perp$. Since $\mathbb{R}^n \subseteq \Sigma$, the vector $x_h := \alpha_1 a_1 + h$ with $\alpha_1 \in \mathbb{R}$, belongs to a c-cone $\langle a_1, \ldots, a_n\rangle$ say. Thus $x_h = \sum_{i=1}^n \lambda_i a_i$, $\lambda_1, \ldots, \lambda_n \geq 0$, and $h = x_h - \alpha_1 a_1 = (\lambda_1 - \alpha_1)a_1 + \sum_{i\geq 2} \lambda_i a_i$, and $h = \bar{h} = \sum_{i\geq 2} \lambda_i \bar{a}_i$ as stated. The same occurs if $x_h \in \langle a_1', a_2, \ldots, a_n\rangle$ since $a_1 \simeq -a_1'$. To prove the converse, let $x \in \mathbb{R}^n$ and decompose $x = \alpha_1 a_1 + h$ with $h \in a_1^\perp$. Thus h belongs to some cone $\langle \bar{a}_2, \ldots, \bar{a}_n\rangle$. Equivalently, there exist $\lambda_i \geq 0$ such that $h = \sum_{i\geq 2} \lambda_i \bar{a}_i$. We have $a_i = \gamma_i a_1 + \bar{a}_i$, $\gamma_i \in \mathbb{R}$, for each i. Therefore

$$\begin{aligned} x &= \alpha_1 a_1 + h = \alpha_1 a_1 + \sum_{i\geq 2} \lambda_i \bar{a}_i = \alpha_1 a_1 + \sum_{i\geq 2} \lambda_i (a_i - \gamma_i a_1) \\ &= \underbrace{\Big(\alpha_1 - \sum_{i\geq 2} \lambda_i \gamma_i\Big)}_{\alpha} a_1 + \sum_{i\geq 2} \lambda_i a_i \end{aligned}$$

If $\alpha \geq 0$ then $x \in \langle a_1, \ldots, a_n\rangle \subseteq \Sigma$ as required. If $\alpha < 0$ then there exists $\alpha' \geq 0$ such that $\alpha a_1 = \alpha' a_1'$ (using $a_1 \simeq -a_1'$) and $x \in \langle a_1', \ldots, a_n\rangle$ proving that $x \in \Sigma$ as well. ☐

Self surrounding (cf. Definition 3) is in fact equivalent to a Q-covering problem in lower dimension.

Proposition 8. *Assume that $a_i \neq 0$ and let $\bar{v}$ denote the orthogonal projection of a vector v onto $a_i^\perp$, the hyperplane orthogonal to a_i. Then a_i is self-surrounded if and only if $\{\bar{a}_1, \bar{a}_1'\} \oplus \cdots \oplus \{\bar{a}_{i-1}, \bar{a}_{i-1}'\} \oplus \{\bar{a}_{i+1}, \bar{a}_{i+1}'\} \oplus \cdots \oplus \{\bar{a}_n, \bar{a}_n'\}$ defines a Q-covering of $a_i^\perp$.*

Proof. Let's fix i to 1. If there exists a neighborhood U of a_1 such that $U \subseteq \Sigma(a_1)$, then its projection $\bar{U}$ onto $a_1^\perp$ contains 0 in its relative interior and must be included by definition of $\Sigma(a_1)$ in $\{\bar{a}_2, \bar{a}_2'\} \oplus \cdots \oplus \{\bar{a}_n, \bar{a}_n'\}$. Any nonzero vector in $a_1^\perp$ has a representative (with respect to $\simeq$) in $\bar{U}$ and is therefore covered. Thus $a_1^\perp \subseteq \{\bar{a}_2, \bar{a}_2'\} \oplus \cdots \oplus \{\bar{a}_n, \bar{a}_n'\}$. For the converse, consider $\Sigma' = \{a_1, -a_1\} \oplus \{a_2, a_2'\} \oplus \cdots \oplus \{a_n, a_n'\}$. By Proposition 7, $\mathbb{R}^n \subseteq \Sigma'$ and therefore a_1 is surrounded with respect to Σ'. Moreover, a_1 can only be self-surrounded (with respect to Σ') since if it belongs to any c-cone in cones$(\Sigma'(-a_1))$ then such a c-cone must be degenerate and therefore it does not effectively contribute in surrounding a_1. Thus there exists a neighborhood U of a_1 such that $U \subset \Sigma'(a_1)$. By definition of Σ', $\Sigma'(a_1) = \Sigma(a_1)$. Thus $U \subset \Sigma(a_1)$ and a_1 is self-surrounded with respect to Σ as desired. □

Remark 3. *In light of Proposition 8, Proposition 7 can be reformulated as follows. Suppose that there exists an index i such that $a_i \neq 0$ and that $\langle a_i, a_i' \rangle$ is flat. Then $\mathbb{R}^n$ is covered if and only if a_i is self surrounded.*

In general, the Q-covering problem does not enjoy an inductive property, that is, if $\Sigma = \mathbb{R}^n$, it is not necessary that a_i is self surrounded for all i. It is worth mentioning that, in [Cottle, 1980], a subclass of Q-matrices, coined *completely Q-matrices*, was introduced in which the inductive nature is preserved.

Similar to self surrounding, lazy surrounding (cf. Definition 4) can be also seen as a Q-covering problem in dimension $n-1$. The next proposition is stated for a_1 for clarity. The statement holds however for any a_i with appropriate changes.

Proposition 9. *Suppose a_1 belongs to the topological interior of a c-cone $C = \langle a_1', a_2, \ldots, a_n \rangle$. Then $\{\bar{a}_1', \bar{a}_2\} \oplus \cdots \oplus \{\bar{a}_1', \bar{a}_n\}$ defines a Q-covering of $a_1^\perp$.*
The converse holds only when $\det(\, a_1 \; a_2 \; \cdots \; a_n \,)\det(\, a_1' \; a_2 \; \cdots \; a_n \,) > 0$. [7]

Proof. Suppose a_1 belongs to the topological interior of a c-cone $C = \langle a_1', a_2, \ldots, a_n \rangle$, then $a_1 \neq 0$ and $0 \in \langle -a_1, a_1', a_2, \ldots, a_n \rangle^\diamond$. By Lemma 1, $\langle -a_1, a_1', a_2, \ldots, a_n \rangle = (-a_1, a_1', a_2, \ldots, a_n) = (a_1', a_2, \ldots, a_n) = \mathbb{R}^n$ (because C is non-degenerate) and

$$\langle -a_1, a_1', a_2, \ldots, a_n \rangle = C \cup \langle -a_1, a_2, \ldots, a_n \rangle \cup \cup_{i=2}^n \langle -a_1, a_1', a_2, \ldots, a_{i-1}, a_{i+1} \ldots, a_n \rangle = \mathbb{R}^n \,.$$

Moreover, by Lemma 3, $C = \langle a_1, \ldots, a_n \rangle \cup \cup_{i=2}^n \langle a_1', a_1, a_2, \ldots, a_{i-1}, a_{i+1} \ldots, a_n \rangle$. Let $\Sigma' = \{a_1, -a_1\} \oplus \{a_1', a_2\} \oplus \cdots \oplus \{a_1', a_n\}$. Then by definition of Σ' one has

$$\Sigma'(a_1) = \langle a_1, \ldots, a_n \rangle \cup \cup_{i=2}^n \langle a_1, a_1', a_2, \ldots, a_{i-1}, a_{i+1} \ldots, a_n \rangle = C$$
$$\Sigma'(-a_1) = \langle -a_1, a_2, \ldots, a_n \rangle \cup \cup_{i=2}^n \langle -a_1, a_1', a_2, \ldots, a_{i-1}, a_{i+1} \ldots, a_n \rangle$$

Therefore, $\Sigma'(a_1) \cup \Sigma'(-a_1) = \mathbb{R}^n$. By Proposition 7, $\{\bar{a}_1', \bar{a}_2\} \oplus \cdots \oplus \{\bar{a}_1', \bar{a}_n\}$ defines a Q-covering of $a_1^\perp$. Conversely, if $\{\bar{a}_1', \bar{a}_2\} \oplus \cdots \oplus \{\bar{a}_1', \bar{a}_n\}$ defines a Q-covering of $a_1^\perp$ then either a_1 or $-a_1$ is in the topological interior of C. To ensure the former, a_1 and a_1' must not be separated by the hyperplane $(a_2, \ldots, a_n)$ which is equivalent to saying that $\det(\, a_1 \; a_2 \; \cdots \; a_n \,)$ and $\det(\, a_1' \; a_2 \; \cdots \; a_n \,)$ are both nonzero and have the same sign (if a_1 and a_1' are separated by the hyperplane $(a_2, \ldots, a_n)$ then surely $a_1 \notin \langle a_1', a_2, \ldots, a_n \rangle$). □

Both Propositions 8 and 9 will be used in Section 5 to arrive at an algebraic characterization of Q-matrices in dimension 3.

4 Q-covering for $n = 3$

A cone in cones$(\Gamma) \setminus$ cones(Σ) must be necessarily rooted at both a_i and a_i' for some indices i. Following [Garcia et al., 1983], when such an index is unique, the cone is called an *almost c-cone*. Assuming $\Gamma = \mathbb{R}^n$, it is clear that Σ is covering if and only if all minimal cones are covered (necessity is obvious and sufficiency is a corollary of Proposition 3). This section investigates in depth the case $n = 3$ with the objective of a complete understanding of when minimal cones are not covered. For this dimension, all cones in cones$(\Gamma) \setminus$ cones(Σ) are almost c-cones and have the form $\langle a_i, a_i', a_j \rangle$, $j \neq i$. Thus for $n = 3$, assuming $\Gamma = \mathbb{R}^3$, Σ is covering if and only if all (full) almost c-cones are covered. This investigation leads to a complete understating of holes for this dimension (Theorem 1) as well as a characterization of the Q-covering problem for $n = 3$ (Theorem 2) which is amenable to an algebraic characterization as we shall see in Section 5. It is worth mentioning that Corollary 1 reduces the Q-covering problem to the surrounding of the vectors in Σ_0. It strengthens [Garcia et al., 1983, Theorem 4.7] by dropping the *strong non-degeneracy* assumption (which requires the non-degeneracy of all c-cones and almost c-cones). We start by a useful sufficient condition for minimal cones to be covered.

Proposition 10. *Let i, j, k denote three distinct indices and let $G = \langle a_i, a_i', a_j \rangle \subset \mathbb{R}^3$ denote a minimal cone. Assume $\langle a_i, a_i' \rangle$ is covered. If $-a_k \notin G$ then $G \subseteq \Sigma$.*

Proof. The proof is by case distinction following a partition of $\mathbb{R}^3$ suggested by G. Namely, $-G = \langle -a_i, -a_i', -a_j \rangle$, the interior of 7 full cones C_i, the relative interior of 9 facets F_i and 3 rays R_i where $C_1 = G^\diamond$, $C_2 = \langle a_i', -a_i, a_j \rangle^\diamond$, $C_3 = \langle -a_i', a_i, a_j \rangle^\diamond$, $C_4 = \langle -a_i, -a_i', a_j \rangle^\diamond$, $C_5 = \langle a_i, a_i', -a_j \rangle^\diamond$, $C_6 = \langle a_i', -a_i, -a_j \rangle^\diamond$, and $C_7 = \langle -a_i', a_i, -a_j \rangle^\diamond$; $F_1 = \langle a_i, a_j \rangle^\diamond$, $F_2 = \langle a_i, a_i' \rangle^\diamond$, $F_3 = \langle a_i', a_j \rangle^\diamond$, $F_4 = \langle -a_i, a_i' \rangle^\diamond$, $F_5 = \langle -a_i, a_j \rangle^\diamond$, $F_6 = \langle -a_i', a_j \rangle^\diamond$, $F_7 = \langle a_i, -a_i' \rangle^\diamond$, $F_8 = \langle a_i, -a_j \rangle^\diamond$, and $F_9 = \langle a_i', -a_j \rangle^\diamond$; $R_1 = \langle a_i \rangle$, $R_2 = \langle a_i' \rangle$, and $R_3 = \langle a_j \rangle$. Cf. Fig. 2. By minimality of G, a_k cannot be in C_1, F_1, F_2, F_3. If $a_k \in C_5, C_6, C_7, F_4, F_7, F_8, F_9, R_1, R_2$, then $G \subseteq \langle a_i, a_i', a_j, a_k \rangle \subseteq \Sigma$ where the last inclusion is a consequence of Proposition 6. If $a_k \in C_4, F_5, F_6, R_3$, then $G \subseteq G' = \langle a_i, a_i', a_k \rangle$. Since $\langle a_i, a_i' \rangle$ is covered, by Proposition 6, a c-face must intersect $\langle a_i, a_i' \rangle$ or $\langle -a_i, a_i' \rangle^\diamond$ or $\langle a_i, -a_i' \rangle^\diamond$ (such a c-face must have either a_j or a_k as generator) and either $G \subseteq \Sigma$ or $G' \subseteq \Sigma$.

[7] In this case, the hyperplane $(a_2, \ldots, a_n)$ is sometimes said to be *reflective*.

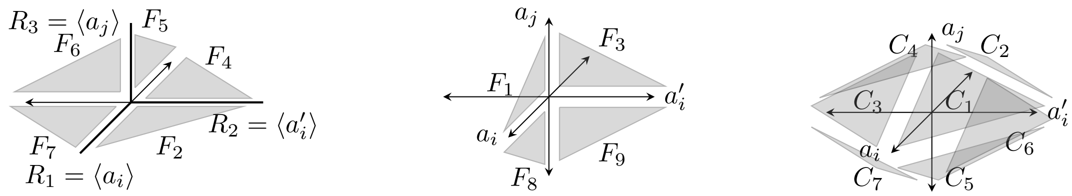


Figure 2: Decomposition of $\mathbb{R}^3$ in the proof of Proposition 10.

In both cases, G is covered. If $a_k \in C_2$ then $G \subseteq \langle a_i, a_j, a_k\rangle \cup G'$. Since $\langle a_i, a_i'\rangle$ is covered, as discussed above, a c-face having either a_j or a_k as generator satisfies the hypothesis of Proposition 6, and either G is covered or G' is covered. Thus, in all cases $G \subseteq \Sigma$. The same holds when $a_k \in C_3$ by swapping a_i and a_i'. ☐

The following proposition is akin to the case of the so called self intersecting 3 *starlike components* [Kelly and Watson, 1979, Fig. 1]. We give below a direct (analytic) proof in line with our purposes.

Proposition 11. *Let $G = \langle a_i, a_i', a_j\rangle$ denote a full cone in $\mathbb{R}^3$ and suppose that there exists a nonzero vector $q \not\simeq a_j$ in $\langle a_i, a_j'\rangle \cap \langle a_i', a_j\rangle$. Let $Q = \langle a_i, a_j, q\rangle$. If $-a_k \in Q$ then $Q^\diamond \subseteq \Sigma(a_k)^c$.*

Proof. For clarity we fix $\{i, j, k\}$ to $\{1, 2, 3\}$. It suffices to permute the indices to match any other configuration. We first observe that Q is non-degenerate because $q \neq 0$, $q \in \langle a_1', a_2\rangle$ and $q \not\simeq a_2$. By hypothesis, (i) $-a_3 = \lambda_1 a_1 + \lambda_2 a_2 + \lambda q$ with $\lambda_1, \lambda_2, \lambda \geq 0$. Moreover, there exist $\gamma_1, \gamma_1' \geq 0$ and $\gamma_2, \gamma_2' > 0$ such that (ii) $q = \gamma_1' a_1' + \gamma_2 a_2$ and (iii) $q = \gamma_1 a_1 + \gamma_2' a_2'$. Using (i) and (ii), one gets $-a_3 = \lambda_1 a_1 + \lambda_2 a_2 + \lambda(\gamma_1' a_1' + \gamma_2 a_2)$ or equivalently $(\lambda_2 + \lambda\gamma_2)a_2 + a_3 = \lambda_1(-a_1) + \lambda\gamma_1'(-a_1')$. Thus $\langle -a_1, -a_1'\rangle \cap \langle a_2, a_3\rangle$ is not reduced to zero. By Lemma 7, $G^\diamond \subseteq (\langle a_1, a_2, a_3\rangle \cup \langle a_1', a_2, a_3\rangle)^c$ and $Q^\diamond \subseteq \Sigma(a_2, a_3)^c$. Moreover, let $x \in \langle a_1, a_2, q\rangle^\diamond$, that is $x = \alpha_1 a_1 + \alpha_2 a_2 + \alpha q$, with $\alpha_1, \alpha_2, \alpha > 0$.

- If $x \in \langle a_1, q, a_3\rangle$ then $x = \beta_1 a_1 + \beta q + \beta_3 a_3 = \alpha_1 a_1 + \alpha_2 a_2 + \alpha q$ with $\beta_1, \beta, \beta_3 \geq 0$. Thus $(\beta - \alpha)q + \beta_3 a_3 = (\alpha_1 - \beta_1)a_1 + \alpha_2 a_2$. If $\beta_3 = 0$ then $\alpha_2 = 0$ because a_1, a_2, q are independent, contradicting $\alpha_2 > 0$. Thus $\beta_3 > 0$ and (iv) $-\beta_3 a_3 = -(\alpha_1 - \beta_1)a_1 - \alpha_2 a_2 + (\beta - \alpha)q = \beta_3\lambda_1 a_1 + \beta_3\lambda_2 a_2 + \beta_3\lambda q$. Using (i) and (iv) one gets $-\alpha_2 = \beta_3\lambda_2$, a contradiction.
- If $x \in \langle a_2', q, a_3\rangle$ then $x = \beta_2' a_2' + \beta q + \beta_3 a_3 = \alpha_1 a_1 + \alpha_2 a_2 + \alpha q$ with $\beta_2', \beta, \beta_3 \geq 0$. Thus $(\beta - \alpha)q + \beta_3 a_3 = \alpha_1 a_1 + \alpha_2 a_2 - \beta_2' a_2'$. If $\beta_3 = 0$. Then $\beta - \alpha$ cannot be zero since a_1, a_2', a_2 are independent ($q \not\simeq a_2$). Moreover, if $\beta - \alpha \neq 0$, $q \in \langle a_1, a_2'\rangle$ forces α_2 to be zero contradicting $\alpha_2 > 0$. So $\beta_3 > 0$ and (v) $-\beta_3 a_3 = -\alpha_1 a_1 - \alpha_2 a_2 + \beta_2' a_2' + (\beta - \alpha)q = \beta_3\lambda_1 a_1 + \beta_3\lambda_2 a_2 + \beta_3\lambda q$. Using (iii), one substitutes a_2' for a linear combination of a_2 and q. Thus (i) and (v) lead to $-\alpha_1 = \beta_3\lambda_1$, a contradiction.

Since $q \in \langle a_1', a_2\rangle$, $\langle a_1, a_2', a_3\rangle = \langle a_1, q, a_3\rangle \cup \langle a_2', q, a_3\rangle$. Therefore $x \notin \langle a_1, a_2', a_3\rangle$. Finally, if $x \in \langle a_1', a_2', a_3\rangle$ then $x = \beta_1' a_1' + \beta_2' a_2' + \beta_3 a_3 = \alpha_1 a_1 + \alpha_2 a_2 + \alpha q$ with $\beta_1', \beta_2', \beta_3 \geq 0$. Using $q = \gamma_1 a_1 + \gamma_2' a_2' = \gamma_1' a_1' + \gamma_2 a_2$, if $\gamma_1' = 0$ then $q \simeq a_2$, contradicting the hypothesis on q. Thus $\gamma_1' > 0$. If $\gamma_2' = 0$ then G would be degenerate, also a contradiction. Thus $\gamma_2' > 0$. One then gets (v) $\alpha_1 a_1 + \alpha_2 a_2 + \alpha q = \frac{\beta_1'}{\gamma_1'}(q - \gamma_2 a_2) + \frac{\beta_2'}{\gamma_2'}(q - \gamma_1 a_1) + \beta_3 a_3$. If $\beta_3 = 0$ then $\alpha_1 + \frac{\beta_2'}{\gamma_2'}\gamma_1 = 0$ contradicting $\alpha_1 > 0$. Thus $\beta_3 > 0$ and using (i) and (v) one gets $-(\alpha_1 + \frac{\beta_2'}{\gamma_2'}\gamma_1) = \beta_3\lambda_1$ which is impossible. Thus $x \notin \langle a_1', a_2', a_3\rangle$ and $Q^\diamond \subseteq \Sigma(a_2', a_3)^c$. Therefore $Q^\diamond \subseteq \Sigma(a_2, a_3)^c \cap \Sigma(a_2', a_3)^c = \Sigma(a_3)^c$. ☐

Proposition 5 showed that the cone $\langle a_i, a_i'\rangle$ enjoys a special property: it cannot be partially covered. When it is covered, Theorem 1 below gives a necessary condition for a minimal cone $\langle a_i, a_i', a_j\rangle$ to contain a hole. Its proof requires the following technical lemma.

Lemma 9. *Let i, j, k denote distinct indices in $\{1, 2, 3\}$, $G = \langle a_i, a_i', a_j\rangle$ denote a full cone in $\mathbb{R}^3$ such that $-a_k \in G$. Suppose there exist vectors $p, q \neq 0$ such that $p \in \langle -a_i, a_i'\rangle^\diamond \cap \langle a_j', a_k\rangle^\diamond$ and $q \in \langle a_i, a_j'\rangle^\diamond \cap \langle a_i', a_j\rangle^\diamond$. Then $-a_k \in \langle a_i, a_j, q\rangle$.*

Proof. For clarity we fix $\{i, j, k\}$ to $\{1, 2, 3\}$. It suffices to permute the indices to match any other configuration. Since $p \in \langle -a_1, a_1'\rangle^\diamond \cap \langle a_2', a_3\rangle^\diamond$, there exist $\lambda_1, \lambda_1' > 0$ and $\lambda_2', \lambda_3 > 0$ such that (i) $p = \lambda_1(-a_1) + \lambda_1' a_1' = \lambda_2' a_2' + \lambda_3 a_3$. Moreover since $q \in \langle a_1, a_2'\rangle^\diamond \cap \langle a_1', a_2\rangle^\diamond$, there exist $\gamma_1', \gamma_2, \gamma_1, \gamma_2' > 0$, such that (ii) $q = \gamma_1' a_1' + \gamma_2 a_2 = \gamma_1 a_1 + \gamma_2' a_2'$. We can eliminate a_2' from (i) and (ii) to get

$$-\gamma_2'\lambda_3 a_3 = \underbrace{(-\lambda_2'\gamma_1 + \gamma_2'\lambda_1)}_{\theta_1} a_1 + \underbrace{(-\gamma_2'\lambda_1' + \lambda_2'\gamma_1')}_{\theta_1'} a_1' + \lambda_2'\gamma_2 a_2$$

with $\theta_1, \theta_1' \geq 0$ (because $-a_3 \in G$). Thus $\frac{\gamma_1'}{\lambda_1'} \geq \frac{\gamma_2'}{\lambda_2'} \geq \frac{\gamma_1}{\lambda_1}$ and $\rho_1 = \gamma_1'\lambda_1 - \lambda_1'\gamma_1 \geq 0$. We further eliminate a_1' from (i) and (ii) to get

$$\rho_1 a_1 + \lambda_1'\gamma_2 a_2 + \theta_1' a_2' = -\gamma_1'\lambda_3 a_3$$

Thus $-a_3 \in G \cap \langle a_1, a_2, a_2'\rangle = \langle a_1, a_2, q\rangle$. ☐

Theorem 1. *Let $G = \langle a_i, a_i', a_j\rangle$, $j \neq i$, denote a minimal cone such that $\langle a_i, a_i'\rangle \subseteq \Sigma$. If G contains a hole K then, up to swapping a_i, a_i', there exists a nonzero vector $q \in \langle a_i, a_j'\rangle \cap \langle a_i', a_j\rangle$ such that $K = \langle a_i, q, a_j\rangle^\diamond$.*

Proof. If $-a_k$ or $-a_k'$ is not in G then, by Proposition 10, $G \subseteq \Sigma$. By hypothesis G contains a hole so we can assume that $-a_k, -a_k' \in G$. Since $\langle a_i, a_i'\rangle$ is covered, by Proposition 6, one c-face must intersect $\langle a_i, a_i'\rangle$ or $\langle -a_i, a_i'\rangle^\diamond$ or $\langle a_i, -a_i'\rangle^\diamond$. In this case, such a c-face must have a_j' as a generator (as both $\langle a_j, a_k\rangle$ and $\langle a_j, a_k'\rangle$ intersect $\langle -a_i, -a_i'\rangle$). With respect to the partition of the space used in the proof of Proposition 10 (cf. Fig 2), and since G is a minimal cone, for $\langle a_i, a_i'\rangle$ to be covered, a_j' must belong to one of the following $C_2, C_3, F_4, F_7, R_1, R_2$. In all cases there exists a nonzero vector $q \not\simeq a_j'$ in $\langle a_i, a_j'\rangle \cap \langle a_i', a_j\rangle$ (when a_j' is in C_2, F_4, R_2) or in $\langle a_i, a_j\rangle \cap \langle a_i', a_j'\rangle$ (when a_j' is in C_3, F_7, R_1). (Observe that both cases are symmetric by swapping a_i and a_i'.) Suppose the former, and let $Q = \langle a_i, a_j, q\rangle$. If $a_j' \in R_2$ or $a_j' \in F_4$ then $q \simeq a_i'$ and $Q = G$. By Proposition 11, $G^\diamond = Q^\diamond \subseteq \Sigma(a_k)^c \cap \Sigma(a_k')^c = \Sigma^c$. Suppose $a_j' \in C_2$, then there exists $p \in \langle -a_i, a_i'\rangle^\diamond \cap \langle a_j', a_k\rangle^\diamond$. By Lemma 9, $-a_k \in Q$ and by Proposition 11, $Q^\diamond \subseteq \Sigma(a_k)^c$. If $\langle a_j', a_k'\rangle$ does not intersect $\langle -a_i, a_i'\rangle^\diamond$ then it intersects $\langle -a_i, -a_i'\rangle$ and therefore $G^\diamond \subseteq \Sigma(a_k')^c$ by Lemma 7. Thus $Q^\diamond \subseteq \Sigma^c$. Otherwise $\langle a_j', a_k'\rangle$ intersect $\langle -a_i, a_i'\rangle^\diamond$ and $-a_k' \in Q$ and by Proposition 11, $Q^\diamond \subseteq \Sigma(a_k')^c$, $Q^\diamond \subseteq \Sigma^c$. As $Q^\diamond$ is the maximal hole contained in G, it follows that $K = Q^\diamond$ as stated. □

Corollary 1. *$\Sigma = \mathbb{R}^3$ if and only if, for all i, both a_i and a_i' are surrounded.* [8]

Proof. Necessity is immediate. For sufficiency, we prove the contrapositive, that is if Σ^c is non-empty then there exists an index i for which either a_i or a_i' is not surrounded. Assume first that $\Gamma \subset \mathbb{R}^3$. Then Γ is a closed proper convex cone of $\mathbb{R}^3$ with a boundary that is non-empty. If for all $a \in \Sigma_0$, $a \in \Gamma^\diamond$ then $\Gamma \subseteq \Gamma^\diamond \subseteq \Gamma$. So Γ is both open and closed and its boundary must be empty, a contradiction. [9] Thus, there exists a vector $a \in \Sigma_0$ such that a is a boundary ray of Γ and therefore a cannot be surrounded (if U is a neighborhood of a, then $U \not\subseteq \Gamma$ and since $\Sigma \subseteq \Gamma$, $U \not\subseteq \Sigma$). Next, assume that $\Gamma = \mathbb{R}^3$. Since $\Sigma \subset \mathbb{R}^3$, by Proposition 3, there must exist a minimal cone $G = \langle a_i, a_i', a_j\rangle$, $j \neq i$, such that $K = G \cap \Sigma^c$ is non-empty. By Proposition 5, either $\langle a_i, a_i'\rangle^\diamond \subseteq \Sigma^c$ or $\langle a_i, a_i'\rangle \subseteq \Sigma$. If the former holds then both a_i and a_i' are not surrounded. If the latter holds, then by Theorem 1, either a_i is not surrounded or a_i' is not surrounded. □

Corollary 1 strengthens [Garcia et al., 1983, Theorem 4.7] by dropping the strong non-degenerate assumption. The latter result was moreover established using degree theory while in this work we solely used convex geometry. The statement of Corollary 1 does not hold for $n > 3$. In [Morris Jr, 1988, Theorem 3], the author gives an example in $n = 4$ where both a_i and a_i' are (lazily) surrounded for all i without having a Q-covering. The (counter)example in $n = 4$ thus shows that it is possible for a minimal cone to have surrounded generators while still having a hole in it.

Another (more practical) issue with the statement of Corollary 1 is that, in general, surrounding is not straightforward to transpose algebraically. Under the assumption $\Gamma = \mathbb{R}^3$, in the sequel we alleviate the need for checking surrounding: we show that self surrounding and lazy covering are enough to characterize the Q-covering for $n = 3$. We start by proving some special cases before stating the main theorem (cf. Theorem 2 below).

Lemma 10. *Suppose that $a_i \in \langle a_i', a_j\rangle$, $i \neq j$. If a_i' is self surrounded or lazily covered then all minimal cones rooted at a_i, a_i' are covered.*

Proof. For clarity we fix (i, j) to $(1, 2)$. (It suffices to permute the indices accordingly for any other configuration.) Since $a_1 \in \langle a_1', a_2\rangle$, then $\langle a_1, a_1', a_2\rangle \subseteq \langle a_1', a_2\rangle \subseteq \Sigma$, $\langle a_1, a_1', a_3\rangle \subseteq \langle a_1', a_2, a_3\rangle \subseteq \Sigma$, and $\langle a_1, a_1', a_3'\rangle \subseteq \langle a_1', a_2, a_3'\rangle \subseteq \Sigma$. Only $G = \langle a_1, a_1', a_2'\rangle$ is left out. Assume a_1' is self surrounded. Thus $\langle a_1, a_1'\rangle$ is covered by Proposition 5. Suppose $-a_3, -a_3' \in G$, then by Proposition 11, $G^\diamond \subseteq \Sigma(a_3)^c \cap \Sigma(a_3')^c = \Sigma^c$, contradicting the self surrounding of a_1'. Thus either $-a_3 \notin G$ or $-a_3' \notin G$ and by Proposition 10, G is covered as desired. Assume $a_1' \in \Sigma(a_1)$. If $a_1' \in \langle a_1, a_2', a_3\rangle$, then $G \subseteq \langle a_1, a_2', a_3\rangle \subseteq \Sigma$. The same occurs if $a_1' \in \langle a_1, a_2', a_3'\rangle$ by swapping a_3 and a_3'. If $a_1' \in \langle a_1, a_2, a_3\rangle$ then either $a_3 \simeq a_1'$ or $a_3 \in \langle -a_1, a_1'\rangle^\diamond$. The former makes G a c-cone. If the latter occurs, then by Lemma 6, G is covered. The same discussion holds if $a_1' \in \langle a_1, a_2, a_3'\rangle$ by swapping a_3 and a_3'. We thus proved that all minimal cones rooted at a_1, a_1' are covered. □

Lemma 11. *Suppose $a_i \in \langle a_j, a_k\rangle^\diamond$ with i, j, k distinct. Then all minimal cones rooted at a_i, a_i' are covered.*

Proof. For clarity we fix i, j, k to $1, 2, 3$ respectively. (It suffices to permute the indices accordingly for any other configuration.) Since $a_1 \in \langle a_2, a_3\rangle$ then $\langle a_1, a_1', a_3\rangle \subseteq \langle a_1', a_2, a_3\rangle \subseteq \Sigma$ and $\langle a_1, a_1', a_2\rangle \subseteq \langle a_1', a_2, a_3\rangle \subseteq \Sigma$. It remains to prove that $\langle a_1, a_1', a_2'\rangle$ and $\langle a_1, a_1', a_3'\rangle$ are covered. We first prove that $G = \langle a_1, a_1', a_2'\rangle$ is covered. If $a_2 \simeq a_2'$, then G is also covered as discussed above. If $a_2 \simeq a_1'$ then $G \subseteq \langle a_1', a_2', a_3\rangle \subseteq \Sigma$. In addition $a_2 \not\simeq a_1$ (because $a_1 \in \langle a_2, a_3\rangle^\diamond$). Using the minimality of G, we can thus assume in the sequel that $a_2 \notin G$. Suppose that $-a_3 \in G$, then there exists $\lambda_1, \lambda_1', \lambda_2' \geq 0$ such that $-a_3 = \lambda_1 a_1 + \lambda_1' a_1' + \lambda_2' a_2'$.

[8] In [Kelly and Watson, 1979], the authors relied heavily on visualization to characterize 3×3 non-degenerate Q-matrices. Interested readers can find a proof in the same spirit in [Kozaily, 2024, Proposition 4].

[9] Another way to arrive at a contradiction would be to use the fact that $\mathbb{R}^n$ is a connected topological space, thus the only clopen subsets are $\mathbb{R}^n$ and its complement, excluding Γ.

Moreover there exists $\gamma_2, \gamma_3 > 0$ such that $a_1 = \gamma_2 a_2 + \gamma_3 a_3$. By eliminating a_3 from both equations, one gets

$$\gamma_2 a_2 = (\gamma_3 \lambda_1 + 1) a_1 + \gamma_3 \lambda'_1 a'_1 + \gamma_3 \lambda'_2 a'_2$$

and therefore $a_2 \in G$, a contradiction. Thus $-a_3 \notin G$. By hypothesis $\langle a_1, a'_1 \rangle \subseteq \langle a'_1, a_2, a_3 \rangle \subseteq \Sigma$. Thus Proposition 10 applies and $G \subseteq \Sigma$. The exact same discussion holds to prove that $\langle a_1, a'_1, a'_3 \rangle$ is covered by swapping the indices 2 and 3. □

Theorem 2. *Assume $\Gamma = \mathbb{R}^3$. Then $\Sigma = \mathbb{R}^3$ if and only if, all vectors in Σ_0 are either self surrounded or lazily covered. (Observe that necessity holds for any finite dimension.)*

Proof. (Necessity) If $\Sigma = \mathbb{R}^3$, then a_i must be surrounded for all i. If a_i is not self surrounded then necessarily $a_i \in \Sigma(a'_i)$ (otherwise a_i cannot be surrounded). The same holds for a'_i. (Sufficiency) We prove the contrapositive, i.e. if $\Sigma \subset \mathbb{R}^3$ then there exists a vector in Σ_0 which is not self surrounded nor lazily covered. The proof is by contradiction. Suppose that $\Sigma \subset \mathbb{R}^3$ and all vectors in Σ_0 are either self surrounded or lazily covered. Thus, there exists a hole K. Since $\Gamma = \mathbb{R}^3$, then by Proposition 3 there exists a minimal cone $G = \langle a_i, a'_i, a_j \rangle$, $j \neq i$, such that $K = G \cap \Sigma^c$ is non-empty. By Proposition 5, either $\langle a_i, a'_i \rangle^\diamond \subset \Sigma^c$ or $\langle a_i, a'_i \rangle \subset \Sigma$. If the former holds, then a_i (or a'_i) leads to a contradiction as it is not self surrounded nor lazily covered (cf. the proof of Proposition 5). If the latter holds, by Theorem 1, a_i, say, is not surrounded (otherwise it suffices to swap a_i and a'_i in what follows). In particular a_i is not self nor lazily surrounded. Therefore $a_i \in \Sigma_2(a'_i)$ (i.e. a face of a c-cone rooted at a'_i). If $a_i \in \langle a'_i, a_j \rangle$, $j \neq i$, then by Lemma 10, G is covered. If $a_i \in \langle a_j, a_k \rangle^\diamond$, with i, j, k distinct, then by Lemma 11, G is also covered. Both cases lead to a contradiction since K is non-empty. Thus K does not exist and $\Sigma = \mathbb{R}^3$. □

Unlike Corollary 1, Theorem 2 is amenable to an *algebraic characterization* of Q-matrices for $n = 3$ as we shall detail in the next section.

5 Algebraic Characterization

Theorem 2 characterizes the Q-covering via three requirements: (1) Feasibility or S-matricity (i.e. $\Gamma = \mathbb{R}^3$), (2) self surrounding (i.e. the Q-covering of $a_i^\perp$), and (3) lazy covering (i.e. $a_i \in \Sigma(a'_i)$). We detail next how each requirement can be equivalently translated into sign conditions on the subdeterminants of the matrix $\left(\begin{smallmatrix} a_1 & a_2 & a_3 & a'_1 & a'_2 & a'_3 \end{smallmatrix} \right)$.

5.1 S-matricity

Proposition 1 provides an effective mean to characterize S-matrices for $n = 3$.

Corollary 2. *Let $\Gamma = \langle g_1, \ldots, g_6 \rangle$. Then $\Gamma = \mathbb{R}^3$ if and only if*

- *either there are 4 vectors such that $\mathbb{R}^3 = \langle g_{i_1}, \ldots, g_{i_4} \rangle$,*
- *or there are 3 vectors such that $\langle g_{i_1}, g_{i_2}, g_{i_3} \rangle$ is a plane that separates two other vectors g_{i_4} and g_{i_5},*
- *or there are 2 vectors such that $\langle g_{i_1}, g_{i_2} \rangle$ is a line and the plane $g_{i_1}^\perp$ is equal to $\langle \pi(g_{i_3}), \ldots, \pi(g_{i_6}) \rangle$, where π denotes the orthogonal projection onto the hyperplane $g_{i_1}^\perp$.*

Proof. Sufficiency is immediate. For necessity, by Proposition 1, there exists $1 \leq m \leq 3$ such that $m + 1$ vectors among $g_1, \ldots, g_6$ span a flat of dimension m. The provided conditions enumerate all cases ($m = 3$ first, then $m = 2$, and finally $m = 1$). □

When the four cones $\langle g_1, g_2, g_3 \rangle$, $\langle g_2, g_3, g_4 \rangle$, $\langle g_1, g_3, g_4 \rangle$, and $\langle g_1, g_2, g_4 \rangle$ are full and have the same orientation, then $\Gamma = \langle g_1, g_2, g_3, g_4 \rangle = \mathbb{R}^3$. The orientation can be retrieved using the sign of the determinant of the matrix formed by the generators. For instance, the orientation of the cone $\langle g_1, g_2, g_3 \rangle$ is given by the sign of $\det\left(\begin{smallmatrix} g_1 & g_2 & g_3 \end{smallmatrix} \right)$. However, while the cone, as a geometric object, is invariant under any permutation of its generators, the sign of the determinant is not. One thus has to be careful when ordering the vectors to get a *coherent orientation* of the involved cones [Jeanneret and Lines, 2014, Section 18]. We do this by fixing a global order of the involved vectors and making sure that the orientation of the common facet of any two adjacent cones is inverted w.r.t. the fixed global order. For instance the cones spanned by the lists $\{g_1, g_2, g_3\}$ and $\{g_2, g_3, g_4\}$ are adjacent cones having in common $\{g_2, g_3\}$. With respect to the global ordering g_1, g_2, g_3, g_4, the lists $\{g_1, g_2, g_3\}$ and $\{g_3, g_2, g_4\}$ are coherently oriented.

Similarly, when $\mathbb{R}^3$ is nonnegatively spanned by five vectors, assuming any 4 of them do not span the space, then three of them span a plane that separates the two remaining ones. This implies that the space is partitioned into six distinct cones that have the same orientation (i.e. all pairs of adjacent cones are coherently oriented). For a list of 5 vectors, we misuse the $\oplus$ notation (cf. Def. 2) and write Γ as $\{g_1, g_2, g_3\} \oplus \{g_4, g_5\}$, where the first three vectors span the plane separating the two remaining ones. Finally, when the six vectors are required to span $\mathbb{R}^3$, they must form 3 lines in a generic position (no one is in the plane formed by the two others). This implies that the space is partitioned into eight distinct cones that have the same orientation. In this case, we write Γ as $\{g_1, g_2\} \oplus \{g_3, g_4\} \oplus \{g_5, g_6\}$ where each pair form a

Algorithm 1: Γ_4: $\mathbb{R}^3 = \langle g_1, g_2, g_3, g_4 \rangle$.

Data: Four symbolic vectors $g_1, \ldots, g_4$.

$\{d_1, d_2, d_3, d_4\} \leftarrow \{\det(g_1\ g_2\ g_3), \det(g_2\ g_1\ g_4), \det(g_3\ g_4\ g_1), \det(g_4\ g_3\ g_2)\}$

return $(\bigwedge_{i=1}^{4} d_i > 0) \vee (\bigwedge_{i=1}^{4} d_i < 0)$

Algorithm 2: Γ_5: $\mathbb{R}^3 = \langle g_1, \ldots, g_5 \rangle = \{g_1, g_2, g_3\} \oplus \{g_4, g_5\}$.

Data: Five symbolic vectors $g_1, \ldots, g_5$.

$\{d_1, d_2, d_3\} \leftarrow \{\det(g_1\ g_2\ g_4), \det(g_2\ g_3\ g_4), \det(g_3\ g_1\ g_4)\}$

$\{d_4, d_5, d_6\} \leftarrow \{\det(g_1\ g_3\ g_5), \det(g_3\ g_2\ g_5), \det(g_2\ g_1\ g_5)\}$

return $(\bigwedge_{i=1}^{6} d_i > 0) \vee (\bigwedge_{i=1}^{6} d_i < 0)$

line. The notation here is suggestive of the Q-covering problem. In fact it is a very special case where one gets a partition of the space. Equivalently, $(g_1 g_3 g_5)^{-1}(g_2 g_4 g_6)$ is a P-matrix.

We say that a vector is *symbolic* if all its components are non-fixed reals or variables or *symbols*. For instance the vector (m_1, m_2, m_3) is symbolic whereas $(1, 0, 0)$ is not. Algorithm 1 makes explicit the conditions that the components of 4 vectors have to satisfy to span $\mathbb{R}^3$. Algorithm 2 makes explicit the conditions that a list of 5 vectors have to satisfy to span $\mathbb{R}^3$ assuming the first three vectors span a plane and the remaining two vectors are separated by that plane. To avoid checking whether the first three vectors actually form a plane, the algorithm relies on partitioning the space into 6 coherently oriented cones. The so obtained conditions may thus have some redundancy with the ones obtained from Algorithm 1. Finally, Algorithm 3 makes explicit the conditions that a list of 6 vectors have to satisfy to span $\mathbb{R}^3$ (where all the vectors are required). As discussed earlier, we also implement the partition of the space into 8 cones for simplicity at the cost of redundancy with the conditions provided by the two other algorithms.

Given 6 symbolic vectors $g_1, \ldots, g_6$, one can characterize $\Gamma = \langle g_1, \ldots, g_6 \rangle = \mathbb{R}^3$ as follows: apply Algorithm 1 to any sublist of four vectors among the ones provided, apply Algorithm 2 to all distinct pairs of sublists of 3 and 2 vectors, and finally apply Algorithm 3 to all distinct tuples formed each by three pairs of vectors.

5.2 Self Surrounding

By Proposition 8, self surrounding for $n = 3$ amounts to characterizing the Q-covering for $n = 2$. The following theorem is the analogue of Corollary 1 for dimension 2. It relies solely on surrounding which is easy to characterize for this dimension, alleviating the need for feasibility.

Theorem 3. *$\Sigma = \mathbb{R}^2$ if and only if, for all i, a_i and a'_i are surrounded.*

Proof. Necessity is immediate. For sufficiency, we prove the contrapositive. Suppose $\Sigma \subset \mathbb{R}^2$. If $\Gamma \subset \mathbb{R}^2$ then by [Rockafellar, 1997, Corollary 18.3.1], there exists a vector in Σ_0 which is a face of Γ. Since $\Sigma \subseteq \Gamma$, such a vector cannot be surrounded. Next, suppose that $\Gamma = \mathbb{R}^2$. By proposition 3, there must exist a minimal cone $G = \langle a_i, a'_i \rangle$ such that $G \cap \Sigma^c$ is non-empty and, by proposition 5, $G^\diamond \subseteq \Sigma^c$ proving that both a_i and a'_i are not surrounded. □

Proposition 12. *Suppose $n = 2$ and let i, j denote two distinct indices. Let $\bar{v}$ denote the orthogonal projection of $v \in \mathbb{R}^2$ onto $a_i^\perp$. Then a_i is surrounded if and only if it is either self or lazily surrounded or $a_i \simeq a_j$ and the pair $\{\bar{a}'_i, \bar{a}'_j\}$ form a 1-dimensional Q-covering or $a_i \simeq a'_j$, and the pair $\{\bar{a}'_i, \bar{a}_j\}$ form a 1-dimensional Q-covering.*

Proof. Sufficiency is immediate using Proposition 8. For necessity, assume that a_i is not self nor lazily surrounded. Thus $a_i \in \Sigma_1(a'_i)$. If $a_i \simeq a'_i$, then Σ reduces to $\langle a_i, a_j \rangle \cup \langle a_i, a'_j \rangle$ and a_i is surrounded if and only if it is self surrounded, contradicting the assumption. So $a_i \not\simeq a'_i$. If $a_i \simeq a_j$ and $a_j \simeq a'_j$ then the surrounding is impossible as Σ reduces to one cone having a_i as generator. If $a_j \not\simeq a'_j$, then a_i must be surrounded by $\langle a_i, a'_j \rangle \cup \langle a'_i, a_j \rangle$ which is effectively equivalent to checking that $\{\bar{a}'_i, \bar{a}'_j\}$ form a 1-dimensional Q-covering by Proposition 8. The same discussion holds when $a_i \simeq a'_j$ and one needs to check that $\{\bar{a}'_i, \bar{a}_j\}$ form a 1-dimensional Q-covering. □

Proposition 12 is amenable to an algebraic characterization of the Q-covering problem in $n = 2$. For lazy surrounding, checking if $a_i \in \langle a'_i, a_j \rangle^\diamond$, $j \neq i$, amounts to simply checking that the determinants of the three matrices $(a'_i \quad a_j)$, $(a'_i \quad a_i)$, and $(a_i \quad a_j)$ have the same sign. Likewise, checking if a_i is self surrounded amounts to verifying that the determinants of $(a_i \quad a_j)$ and $(a_i \quad a'_j)$, $j \neq i$, have opposite signs. We observe that this condition should not be confused with $a_i \in \langle a_j, a'_j \rangle^\diamond$, which is only a special case (a_i needs not be in the interior of $\langle a_j, a'_j \rangle$ to be self surrounded). As stated in Proposition 8, self surrounding can be equivalently checked by projecting on the orthogonal space of a_i and appealing to the following simple fact.

Algorithm 3: Γ_6: $\mathbb{R}^3 = \langle g_1, \ldots, g_6 \rangle = \{g_1, g_2\} \oplus \{g_3, g_4\} \oplus \{g_5, g_6\}$.

Data: Six symbolic vectors $g_1, \ldots, g_6$.

$\{d_1, d_2, d_3, d_4\} \leftarrow \{\det(\begin{smallmatrix} g_1 & g_3 & g_5 \end{smallmatrix}), \det(\begin{smallmatrix} g_3 & g_2 & g_5 \end{smallmatrix}), \det(\begin{smallmatrix} g_2 & g_4 & g_5 \end{smallmatrix}), \det(\begin{smallmatrix} g_4 & g_1 & g_5 \end{smallmatrix})\}$

$\{d_5, d_6, d_7, d_8\} \leftarrow \{\det(\begin{smallmatrix} g_1 & g_4 & g_6 \end{smallmatrix}), \det(\begin{smallmatrix} g_4 & g_2 & g_6 \end{smallmatrix}), \det(\begin{smallmatrix} g_2 & g_3 & g_6 \end{smallmatrix}), \det(\begin{smallmatrix} g_3 & g_1 & g_6 \end{smallmatrix})\}$

return $(\bigwedge_{i=1}^{8} d_i > 0) \vee (\bigwedge_{i=1}^{8} d_i < 0)$

Algorithm 4: Surrounding of a_i $(n = 2)$.

Data: Two pairs $\{a_1, a'_1\}$ $\{a_2, a'_2\}$ of vectors in $\mathbb{R}^2$.

1 $\{u_1, u_2\} \leftarrow a_i$

2 $u^{\perp} \leftarrow \{-u_2, u_1\}$ ▷ Orthogonal vector if $a_i \neq 0$

3 $c_1 \leftarrow (u^{\perp}.a_j)(u^{\perp}.a'_j) < 0$ ▷ Self surrounding

4 $c_2 \leftarrow \det(\begin{smallmatrix} a'_i & a_i \end{smallmatrix})\det(\begin{smallmatrix} a_i & a_j \end{smallmatrix}) > 0 \vee \det(\begin{smallmatrix} a'_i & a_i \end{smallmatrix})\det(\begin{smallmatrix} a_i & a'_j \end{smallmatrix}) > 0$ ▷ Lazy surrounding

5 $c_3 \leftarrow \det(\begin{smallmatrix} a_i & a_j \end{smallmatrix}) = 0 \wedge a_i.a_j > 0 \wedge (u^{\perp}.a'_i)(u^{\perp}.a'_j) < 0$ ▷ $a_i \simeq a_j$

6 $c'_3 \leftarrow \det(\begin{smallmatrix} a_i & a'_j \end{smallmatrix}) = 0 \wedge a_i.a'_j > 0 \wedge (u^{\perp}.a'_i)(u^{\perp}.a_j) < 0$ ▷ $a_i \simeq a'_j$

7 **return** $c_1 \vee c_2 \vee c_3 \vee c'_3$

Theorem 4. *Let $a, a' \in \mathbb{R}$. The pair $\{a, a'\}$ defines a Q-covering of $\mathbb{R}$ if and only if $aa' < 0$, providing thereby a partition for $\mathbb{R}$.*

Proof. The cones generated by a and a' cover, or more precisely partition, $\mathbb{R}$ if and only if a, a' are both nonzero and have opposite signs. □

Remark 4. *Observe that, for $n = 1$, Γ and Σ coincide and that a is surrounded if and only if $a \neq 0$. Interestingly, Corollary 1 does not hold for $n = 1$ since a and a' can be both (lazily) surrounded (for instance when $a' \simeq a$) and $\Sigma = \Gamma \subset \mathbb{R}$. Intuitively, when a'_i is lazily surrounded, it is somehow 'redundant' with a_i (with $a'_i \simeq a_i$ being the simplest–and perhaps strongest–form of redundancy). So when a'_i is redundant for all i, Σ is unlikely to be covering.*

Algorithm 4 outputs the set of conditions required for a_i to be surrounded according to Proposition 12. When a_i is identically zero, all conditions fail as desired ($a_i = 0$ cannot be surrounded). Applying the algorithm to the four involved vectors outputs an algebraic characterization for Σ to be covering for $n = 2$; we will denote it in the sequel by `QCovering`$[\{a_1, a'_1\}, \{a_2, a'_2\}]$. In particular, one gets the following characterization for Q-matrices in dimension 2.

Theorem 5. *The matrix $\left(\begin{smallmatrix} m_1 & m_2 \\ m_3 & m_4 \end{smallmatrix}\right)$ is a Q-matrix if and only if*

$$\begin{aligned}
&(m_1 < 0 \wedge m_2 > 0 \wedge m_3 > 0 \wedge m_4 < 0 \wedge m_1 m_4 - m_2 m_3 < 0) \\
\vee\, &(m_1 < 0 \wedge m_2 > 0 \wedge m_3 < 0 \wedge m_4 > 0 \wedge m_1 m_4 - m_2 m_3 > 0) \\
\vee\, &(m_1 = 0 \wedge m_2 > 0 \wedge m_3 < 0 \wedge m_4 > 0) \\
\vee\, &(m_1 > 0 \wedge m_3 = 0 \wedge m_4 > 0) \\
\vee\, &(m_1 > 0 \wedge m_2 \geq 0 \wedge m_4 > 0) \\
\vee\, &(m_1 > 0 \wedge m_2 < 0 \wedge m_3 > 0 \wedge m_1 m_4 - m_2 m_3 > 0) \\
\vee\, &(m_1 > 0 \wedge m_2 < 0 \wedge m_3 < 0 \wedge m_1 m_4 - m_2 m_3 > 0) \ .
\end{aligned} \tag{3}$$

For the sake of comparison, we give below the relatively much simpler conditions for M to be a P-matrix requiring that all the principal minors of M to be positive:

$$m_1 > 0 \wedge m_4 > 0 \wedge m_1 m_4 - m_2 m_3 > 0 \ .$$

One observes that P-matrices are a special case of Q-matrices since $m_1 > 0 \wedge m_4 > 0 \wedge m_1 m_4 - m_2 m_3 > 0$ implies (without being equivalent to) the last four conjunctions of (3).

Remark 5. *The fact that Theorem 5 involves only sign conditions on the subdeterminants of the matrix M is not a coincidence. In fact, Algorithm 4 can be equivalently stated in terms of sign conditions of the subdeterminants of the matrix $(\begin{smallmatrix} a_1 & a_2 & a'_1 & a'_2 \end{smallmatrix})$. Indeed, on one hand, the scalar product $u^{\perp}.a_j$ in Line 3 is equal to $\det(\begin{smallmatrix} a_i & a_j \end{smallmatrix})$. The same holds for the other scalar products involving $u^{\perp}$. On the other hand, for $v, w \in \mathbb{R}^2$, the equivalence $v \simeq w$ characterized by $\det(\begin{smallmatrix} v & w \end{smallmatrix}) = 0 \wedge v.w > 0$ in Algorithm 4, can be reformulated as*

$$(\det(\begin{smallmatrix} v & w \end{smallmatrix}) = 0) \wedge (v_1 w_1 > 0 \vee v_2 w_2 > 0), \tag{4}$$

making explicit the subdeterminants. Therefore, the conditions c_3 (Line 5) and c'_3 (Line 6) are also amenable to sign conditions on appropriate subdeterminants.

Algorithm 5: `SelfSurrounding`: self surrounding of a_i ($n=3$).

Data: Three pairs $\{a_1,a_1'\}$ $\{a_2,a_2'\}$ $\{a_3,a_3'\}$ of vectors in $\mathbb{R}^3$.

$\{u_1,u_2,u_3\} \leftarrow a_i$

$c \leftarrow \pi_u = \begin{pmatrix} u_2 & -u_1 & 0 \\ u_3 & 0 & -u_1 \end{pmatrix} \vee \pi_u = \begin{pmatrix} u_2 & -u_1 & 0 \\ 0 & u_3 & -u_2 \end{pmatrix} \vee \pi_u = \begin{pmatrix} 0 & u_3 & -u_2 \\ u_3 & 0 & -u_1 \end{pmatrix}$

return $c \wedge \mathit{QCovering}[\{\pi_u a_j, \pi_u a_j'\}, \{\pi_u a_k, \pi_u a_k'\}]$

To get an algebraic characterization for self surrounding for $n=3$, in addition to the aforementioned characterization of the Q-covering problem for $n=2$, we further need means to perform the projection on the orthogonal space of a vector $u=(u_1,u_2,u_3)\neq 0$. To do so, we use one of the following generic projectors:

$$\pi_u = \begin{pmatrix} u_2 & -u_1 & 0 \\ u_3 & 0 & -u_1 \end{pmatrix}, \text{ or } \pi_u = \begin{pmatrix} -u_2 & u_1 & 0 \\ 0 & u_3 & -u_2 \end{pmatrix}, \text{ or } \pi_u = \begin{pmatrix} 0 & -u_3 & u_2 \\ -u_3 & 0 & u_1 \end{pmatrix}. \quad (5)$$

Setting u to a_i, one gets that a_i is self surrounded if and only if $\mathbb{R}^2 \subseteq \{\pi_u a_j, \pi_u a_j'\} \oplus \{\pi_u a_k, \pi_u a_k'\}$ as shown in Algorithm 5. Notice that, when $u=0$, $\pi_u=0$ and the algorithm returns False as expected (the vector 0 cannot be self surrounded).

Lemma 12. *Let $u,v,w\in\mathbb{R}^3$ and let π_u denote a generic projector (cf. Eq. (5)). Deciding the sign of $\det(\pi_u v \;\; \pi_u w)$ and the equivalence $\pi_u v \simeq \pi_u w$ reduce to sign conditions on the subdeterminants of the matrix $(u\;v\;w)$.*

Proof. Suppose (a similar discussion holds for the other projectors)

$$\pi_u = \begin{pmatrix} u_2 & -u_1 & 0 \\ u_3 & 0 & -u_1 \end{pmatrix}, \qquad \begin{aligned} s_1 &= u_2v_1 - u_1v_2 \\ s_2 &= u_3v_1 - u_1v_3 \end{aligned} \qquad \begin{aligned} s_3 &= u_2w_1 - u_1w_2 \\ s_4 &= u_3w_1 - u_1w_3 \end{aligned}$$

Then, one has

$$\pi_u v = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}, \quad \pi_u w = \begin{pmatrix} s_3 \\ s_4 \end{pmatrix}, \quad \det(\pi_u v \;\; \pi_u w) = u_1 \det(u\;v\;w),$$

Thus, as mentioned in Remark 5, Eq. (4), the condition

$$\det(\pi_u v \;\; \pi_u w) = 0 \;\wedge\; (\pi_u v).(\pi_u w) > 0,$$

becomes equivalent to

$$u_1 \det(u\;v\;w) = 0 \;\wedge\; (s_1s_3 > 0 \vee s_2s_4 > 0),$$

making therefore explicit the sign conditions on the subdeterminants of $(u\;v\;w)$. ☐

As already observed, self surrounding for $n=3$ reduces to a planar Q-covering problem (Proposition 8) which is in turn equivalent to four surrounding problems for $n=2$ (Theorem 3), each characterized in Proposition 12, and implemented in Algorithm 4. Lemma 12 is the last ingredient to show that self surrounding for $n=3$ reduces to sign conditions on the subdeterminants of the matrix $(a_1\;a_2\;a_3\;a_1'\;a_2'\;a_3')$.

5.3 Lazy Covering

To get an algebraic characterization for the Q-covering problem for $n=3$, we still need to translate the condition $a_i \in \Sigma(a_i')$ into an equivalent explicit set of constraints on the involved vectors. This task reduces to checking whether a vector belongs to a cone spanned by three vectors. First, the equivalence relation $u \simeq v$ is encoded as $u \times v = 0 \wedge u.v > 0$ where $u \times v$ denotes the cross product. To check whether $u \in \langle a_1, a_2, a_3\rangle^\diamond$, we simply verify that $\det(a_1 \;\; a_2 \;\; a_3), \det(u \;\; a_2 \;\; a_3), \det(a_1 \;\; u \;\; a_3), \det(a_1 \;\; a_2 \;\; u)$ are all positive or all negative (the ordering of the column vectors of the matrices ensure the coherence of the orientation; cf. the discussion at the beginning of Section 5); We can alternatively regard lazy surrounding as a Q-covering problem as stated in Proposition 9. It remains to check whether u belongs to the faces of the cone, namely cones of the form $\langle v, w\rangle$. We do so by checking whether u is equivalent to the generators, v, w, or it belongs to the relative interior of $\langle v, w\rangle$ as detailed in Algorithm 6. Finally, Algorithm 7 returns an equivalent characterization for a_i' to be in a c-cone rooted at a_i.

Remark 6. *The conditions of Algorithm 6 are also amenable to equivalent conditions on the sign of the subdeterminants of the matrix $(u\;v\;w)$. For $u \simeq v$, the components of the cross product are subdeterminants by definition. Moreover, the condition $u \times v = 0 \wedge u.v > 0$ is equivalent to $u \times v = 0 \wedge (u_1v_1 > 0 \vee u_2v_2 > 0 \vee u_3v_3 > 0)$, making explicit the subdeterminants. For condition c_3 (Line 3), one first observes that $\det(v\;w\;v{\times}w) > 0$ is equivalent to $v \times w \neq 0$.* [10] *When in addition $\det(u\;v\;w) = 0$, $u = \alpha v + \beta w$ for some scalars α, β, and $\det(v\;u\;v{\times}w) = \beta \det(v\;w\;v{\times}w)$ (the determinant is multilinear and alternating). Thus, assuming $v \times w \neq 0$, $\det(v\;u\;v{\times}w) > 0$ if and only if $\beta > 0$. As $u.v^\perp = \beta w.v^\perp$, one can then reformulate $\beta > 0$ as $\pi_v u \simeq \pi_v w$, which has been shown in Lemma 12 to reduce to sign conditions on the subdeterminants of the matrix $(u\;v\;w)$. The same discussion holds for $\det(u\;w\;v{\times}w) > 0$. Summing up, condition c_3 of Algorithm 6 becomes equivalent to $\det(u\;v\;w) = 0 \wedge v \times w \neq 0 \wedge \pi_v u \simeq \pi_v w \wedge \pi_w v \simeq \pi_w u$.*

Algorithm 6: `InFace`: symbolic characterization of $u \in \langle v, w \rangle$ $(n = 3)$.

Data: Three symbolic vectors of dimension 3.

$c_1 \leftarrow u \times v = 0 \wedge u.v > 0$ ▷ $u \simeq v$
$c_2 \leftarrow u \times w = 0 \wedge u.w > 0$ ▷ $u \simeq w$
$c_3 \leftarrow \det(\,u\ v\ w\,) = 0 \wedge \det(\,v\ w\ v{\times}w\,) > 0 \wedge \det(\,v\ u\ v{\times}w\,) > 0 \wedge \det(\,u\ w\ v{\times}w\,) > 0$ ▷ $u \in \langle v, w \rangle^{\diamond}$
return $(u = 0) \vee c_1 \vee c_2 \vee c_3$

Algorithm 7: `InCone`: symbolic characterization of $a'_i \in \langle a_i, a_j, a_k \rangle$ $(n = 3)$.

Data: Four symbolic vectors of dimension 3.

$\{d_1, d_2, d_3, d_4\} \leftarrow \{\det(\,a_i\ a_j\ a_k\,), \det(\,a'_i\ a_j\ a_k\,), \det(\,a_i\ a'_i\ a_k\,), \det(\,a_i\ a_j\ a'_i\,)\}$
$c_1 \leftarrow \left(\bigwedge_{i=1}^{4} d_i > 0\right) \vee \left(\bigwedge_{i=1}^{4} d_i < 0\right)$ ▷ Topological interior
$c_2 \leftarrow$ `InFace`$[a'_i, \{a_i, a_j\}] \vee$ `InFace`$[a'_i, \{a_i, a_k\}] \vee$ `InFace`$[a'_i, \{a_j, a_k\}]$ ▷ Faces
return $(a'_i = 0) \vee c_1 \vee c_2$

As an immediate corollary of Remark 5, Lemma 12, and Remark 6, we get the following nontrivial result which, until now, was considered an open problem to the best of our knowledge.

Theorem 6. *For $n = 3$, the Q-covering is characterized by sign conditions on the subdeterminants of the matrix $(\,a_1\ a_2\ a_3\ a'_1\ a'_2\ a'_3\,)$. In particular, deciding if a 3-by-3 matrix M is a Q-matrix reduces to sign conditions on the subdeterminants of M effectively constructed by Algorithms 1–7.*

It's worth mentioning, however, that [Garcia et al., 1983] showed that for *super-regular* matrices (cf. Section 5.4), when the *conical degree* of the piecewise linear mapping associated to M is not zero, M is Q-matrix. While the conical degree is determined using the signs of the subdeterminants of M, it was unclear whether the signs of subdeterminants were enough for the cases where the conical degree is zero, and more generally, for matrices that are not super-regular (for which the concept of conical degree is not well defined).

All algorithms were implemented to arrive at an algebraic characterization of the Q-covering problem when $n = 3$. [11] When $\Sigma_0 = \{e_1, e_2, e_3, -M_1, -M_2, -M_3\}$ for a matrix $M = (\,M_1\ M_2\ M_3\,)$, one gets a list of sign conditions on the subdeterminants of M for M to be a Q-matrix. Checking if a given instance is a Q-matrix is thus performed in constant time: it suffices to substitute the values and check the conditions. For instance, for

$$M = \begin{pmatrix} m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 \\ m_7 & m_8 & m_9 \end{pmatrix}, \qquad (6)$$

any specialization of $m_1, \ldots, m_9$ that satisfies

$$m_1 > 0 \wedge m_2 = 0 \wedge m_3 = 0 \wedge m_5 < 0 \wedge m_6 > 0 \wedge m_8 > 0 \wedge m_9 < 0 \wedge m_5 m_9 - m_6 m_8 < 0$$

is a Q-matrix (which is clearly not a P-matrix as $m_5 < 0$). Such a characterization turned out to be very useful to automatically find counter examples to sharpen our intuitions and help answering certain conjectures as we shall see next.

5.4 Generating Special Q-matrices

[Karamardian, 1972] drew a specific attention to R_0 matrices (also known as super-regular matrices) for which LCP$(0, M)$ has a unique solution. It subsequently played an important role in LCP theory.
[Aganagic and Cottle, 1979] proved later that among P_0 matrices (i.e. matrices for which all principal minors are nonnegative), the subclasses of Q-matrices and R_0-matrices are equivalent.

We wanted to know whether a Q-matrix that is not super-regular exists for $n = 3$. It was previously known [Kelly and Watson, 1979, p. 177] that Q-matrices with flat c-cones are possible for $n = 3$. We were thus interested in finding a Q-matrix with non-pointed and non-flat c-cones. To do so we fixed a'_2 to $-e_1$ and found that the following conditions (among others) on the subdeterminants of M (cf. Eq. (6)) have to hold

$$m_1 > 0 \wedge m_3 < 0 \wedge m_6 < 0 \wedge m_7 > 0 \wedge m_9 < 0 \wedge m_1 m_9 - m_3 m_7 > 0 \wedge m_4 m_9 - m_6 m_7 < 0.$$

The following instance is a particular case.

[10] Indeed, $\det(\,v\ w\ v{\times}w\,) = \|v \times w\|^2$ for any vectors v, w. Requiring the strict inequality ensures that the relative interior $\langle v, w \rangle^{\diamond}$ is not empty.

[11] We used Mathematica. Notebook available here https://gitlab.inria.fr/kghorbal/qmatrices.

Example 1. *Consider* $-M_1 = (-2,-4,-3)$, $-M_2 = -e_1$, *and* $-M_3 = (1,1,1)$. *The c-cones* $\langle e_1, -M_2, e_3\rangle$ *and* $\langle e_1, -M_2, -M_3\rangle$ *are non-pointed and therefore the following* 3*-by-*3 *matrix* M *is not super-regular. It is however a Q-matrix.* [12]

$$M = \begin{pmatrix} M_1 & M_2 & M_3 \end{pmatrix} = \begin{pmatrix} 2 & 1 & -1 \\ 4 & 0 & -1 \\ 3 & 0 & -1 \end{pmatrix} .$$

We can also easily check that there are no Q-matrices of the form $\left(v\ e_1\ e_2 \right)$ *for any vector* v *(which would have lead to* 4 *degenerate c-cones that are non-pointed and non-flat).*

[Murty, 1972] gave the symmetric matrix M_{Murty} (see below) for which all vectors in Σ_0 are both self and lazily surrounded. Using the algebraic characterization presented in this work, we checked that when all vectors are lazily surrounded, then they are necessarily also self surrounded (the converse isn't true, it suffices to consider any P-matrix). Enforcing lazy surrounding of all vectors in Σ_0 is unrelated to the symmetry of the matrix M. It turns out that Murty's example is an instance of the following conjunction:

$$m_1 < 0 \wedge m_3 > 0 \wedge m_4 > 0 \wedge m_5 < 0 \wedge m_6 > 0 \wedge m_7 > 0 \wedge m_8 > 0 \wedge m_9 < 0 \\ \wedge m_1 m_5 - m_2 m_4 < 0 \wedge m_1 m_9 - m_3 m_7 < 0 \wedge m_5 m_9 - m_6 m_8 < 0 \ . \quad (7)$$

We give below several non symmetric instances.

Example 2. *The following two matrices are Q-matrices for which all vectors in* Σ_0 *are both self and lazily surrounded (the space is covered twice by the c-cones). They both satisfy the conditions of Eq. (7).*

$$M_{Murty} = \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \qquad M = \begin{pmatrix} -5 & 4 & 3 \\ 2 & -1 & 1 \\ 2 & 2 & -1 \end{pmatrix} .$$

Moreover, there are precisely 3 *additional conjunctions, each involving strict sign conditions on the subdeterminants of* M*, such that all vectors in* Σ_0 *are both self and lazily surrounded. We give below one instance for each such conjunction (the sign conditions could be retrieved from the examples).*

$$\begin{pmatrix} -7 & 5 & 1 \\ -6 & 4 & 1 \\ -8 & 8 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 3 & -9 & 1 \\ 4 & -10 & 1 \\ 16 & -16 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 7 & 5 & -1 \\ 12 & 2 & -1 \\ 8 & 4 & -1 \end{pmatrix} .$$

Conclusion

We believe that using minimal cones to better understand holes is worth pursuing. An important question with this regard is whether the particular case of almost c-cones is all one needs in dimensions ≥ 4. The approach is also appealing as it doesn't require particular assumptions on degeneracy (but does assume feasibility). It thus provides an interesting geometric alternative to degree theory. For $n \leq 3$, one gets in addition an algebraic characterization involving only the signs of the subdeterminants of the involved matrix. So far, however, no particular *pattern* emerged from these conditions (unlike the elegant characterization for P-matrices requiring only the positivity of principal minors). From a computational standpoint, although getting an algebraic characterization was relatively easy (the presented algorithms are straightforward to implement), rewriting the so obtained condition into conjunctions on the signs of subdeterminants was computationally involved as many subcases are either redundant or empty. It would be really interesting to try to push the same reasoning for dimension $n = 4$ to get yet an additional hint about the polynomials involved as well as potential patterns on their sign conditions. We do believe that algebraic characterizations are really helpful to sharpen our intuitions and avoid pitfalls by automatically generating instances with the help of a computer.

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[12] The fact that the provided matrix is a Q-matrix can be checked independently using for instance a quantifier elimination procedure over the reals, like the Cylindrical Algebraic Decomposition or Gale's algorithm [Aganagić and Cottle, 1978, p. 4].

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