Exam 2021-2022

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Pick up at least two out of the three proposed problems below according to your taste. If you enjoyed the mindset of Category theory, check out Problem 1, it has it all. If, however, you don't really care whether every monad arises from an adjunction, Problem 2 might be more suited as it manipulates deductive systems: all you need is a firm logician hat. Finally, if you are more language-oriented, you can chew on the (untyped) constructions of Problem 3. This being said, keep in mind that the three suggested problems are part of the *same story*. The more you appreciate this fact, the more enlightened and powerful you will be, regardless of the next steps of your curriculum.

Stay focused. Clear your thoughts. Enjoy the dive.

1 Functional Completeness as a Universality Property

A *comonad* on a category $\mathscr{A}$ is a monad in the opposite category $\mathscr{A}^{op}$, that is a cotriple (S, ε, δ), where $S : \mathscr{A} \to \mathscr{A}$ is a functor equipped with a counit and a co-multiplication satisfying the associativity and identity laws (needless to say that ε and δ are natural transformations). As we did for monads, we can define the Kleisli category $\mathscr{A}_S$ of a comonad (S, ε, δ) on $\mathscr{A}$ with morphisms $A \to B$ in $\mathscr{A}_S$ whenever $SA \to B$ is a morphism in $\mathscr{A}$ (everything else is like we've seen but with inverting the arrows).

1. State explicitly the associativity and identity laws for a comonad (three diagrams are expected.)

2. State explicitly the identity arrows and composition of morphisms in $\mathscr{A}_S$ (a diagram is expected for the composition).

Let $\mathscr{A}$ be a Cartesian category where $\pi_{A,B}$ and $\pi'_{A,B}$ denote the projections out of the product $A \times B$ of $A, B \in \mathscr{A}$. We use $\langle f, g \rangle$ to denote the *pairing* of f and g, that is the unique map $C \to A \times B$ where $f : C \to A$ and $g : C \to B$.

For any object A in $\mathscr{A}$, define $S_A := A \times -$, $\quad \varepsilon_A(B) := \pi'_{A,B}$, $\quad \delta_A(B) := \langle \pi_{A,B}, 1_{A \times B} \rangle$ (for clarity, we used $\varepsilon_A(B)$ and $\delta_A(B)$ instead of $(\varepsilon_A)_B$ and $(\delta_A)_B$ to avoid double subscripts).

3. Show that $(S_A, \varepsilon_A, \delta_A)$ defines a cotriple of $\mathscr{A}$. We will denote $\mathscr{A}_{S_A}$, or simply $\mathscr{A}_A$, the Kleisli category of the comonad $(S_A, \varepsilon_A, \delta_A)$ on $\mathscr{A}$.

Let $\mathscr{A}[x]$ denote the polynomial category defined over $\mathscr{A}$ assuming the undetermined $x : A_0 \to A$. Let $H_x : \mathscr{A} \to \mathscr{A}[x]$ denote the Cartesian functor that sends $f : A \to B$ onto the constant polynomial with the same name in $\mathscr{A}[x]$ (in words, H_x defines a trivial injection that regards constants as polynomials). The functional completeness can be rephrased as the following *universality property* for H_x. Given a Cartesian category $\mathscr{A}$, any Cartesian functor $F : \mathscr{A} \to \mathscr{B}$ and any arrow $y : F(A_0) \to F(A)$ in $\mathscr{B}$, there exists a unique Cartesian functor $F' : \mathscr{A}[x] \to \mathscr{B}$ such that $F'(x) = y$ and $F' \circ H_x = F' H_x = F$. (The proof of this statement is very similar to—if not the same as—the proof of the deductive theorem deconstructed in Section 2). We will use this universality property to show (with elegance) that the polynomial category $\mathscr{A}[x]$ is isomorphic to the Kleisly category $\mathscr{A}_A$ (A and x are indeed related since A is the codomain of x).

4. Show that $\mathscr{A}_A$ is a Cartesian category. (You need to show that a product exists in $\mathscr{A}_A$, and a product is not a mere isolated object, that is you need to explicit the projections and maps to the terminal object—as a special case of the product of zero elements).

Define the functor $H_A : \mathscr{A} \to \mathscr{A}_A$ by $H_A(B) = B$ and $H_A(f) = f\pi'_{A,C}$ for objects B and arrows $f : C \to B$.

5. Check that H_A is a Cartesian functor.

6. Assume that H_A enjoys the same universality property of H_x with $\pi_{A,1}$ as the undetermined x. Prove that $\mathscr{A}[x]$ is isomorphic to $\mathscr{A}_A$. (Even if you know nothing about polynomial categories and monads, you should be able to prove this just by exploiting universality.)

Bonus (prove that H_A has indeeded the assumed universality property). Let $F : \mathscr{A} \to \mathscr{B}$ denote a Cartesian functor and $y : 1 \to F(A)$ a given arrow in $\mathscr{B}$. We show constructively the existence of a unique Cartesian functor $F' : \mathscr{A}_A \to \mathscr{B}$ that satisfies the desired properties, that is $F' H_A = F$ and $F'(\pi_{A,a}) = y$. Let F' be defined on objects and arrows as $F'(B) = FB$ and $F'(f) = F(f)\langle yF(B)\bullet, 1_{F(B)} \rangle$, where $F(B)\bullet : F(B) \to 1$.

7a. Check that F' is Cartesian.

7b. Check that it satisfies the desired properties.

7c. Prove uniqueness.

2 Deduction Theorem

The standard (and simpler) form of the *deduction theorem* asserts that if $A \wedge B \to C$ then $A \to (C \Leftarrow B)$ (you probably already encountered a similar statement where arrows are denoted by $\vdash$, reads "entails"). However, as soon as one adjoins an assumption $x : \mathbf{T} \to A$ (that is a proof x for the formula A), one obtains a new deductive system $\mathscr{D}(x)$ on which we stated the general form of the deduction theorem. In what follows, you will be guided to prove the theorem for positive intuitionistic propositional calculus. (This is a very general scheme for many proofs in formal languages and abstract algebras.)

1. You may (or may not!) have noticed a circular argument in introducing the 'if' operator $\Leftarrow$ since we also used a sort of 'if ... then ... ' construction (via the inference rule) to introduce $\Leftarrow$. (This is sometimes called the Zeno paradox of logic). How did we solve this issue?

2. What are the (three) primitive operators on proofs in $\mathscr{D}(x)$?

3. Let $\varphi(x)$ denote a proof $B \to C$ in $\mathscr{D}(x)$ where $x : \mathbf{T} \to A$. Deconstruct $\varphi(x)$ using pairing and do the same for transposition. (Bonus: make explicit the five possible forms for $\varphi(x)$).

4. For pairing and transposition, construct an explicit proof $f(x) : A \wedge B \to C$. Here is an example: if $p : B \to C$ is a proof in $\mathscr{D}$ (which is therefore independent from x), then $f = p \circ \pi'_{A,B}$ (which is also independent from x). We can also write f using the λ-abstraction $\lambda_{x:A} p$. Feel free to use similar notations.

5. Observe that $f(x)$, through the primitive operators, deconstructs $\varphi(x)$ into *shorter* proofs. We can make this intuition precise by defining an inductive notion of length on proofs: for instance if $p : B \to C$ exists already in $\mathscr{D}$, then $\varphi(x)$ has length 0 (the constant polynomial). What is the length of a proof pairing two proofs of lengths $n, m \geq 0$?

6. Sketch a proof by induction for the deduction theorem (in $\mathscr{D}(x)$) (outline the main steps based on your previous answers).

7. Bonus: What is the missing ingredient in order to state *functional completeness* in the corresponding Cartesian closed category of $\mathscr{D}(x)$?

3 Church's Numerals

Recall that the untyped λ-calculus is defined inductively by $t ::= x \mid t \wr t' \mid \lambda_x.t$. For simplicity, we will use concatenation to encode $\wr$, so that we write tt' for $t \wr t'$. Church defined some sort of natural numbers, called *numerals*, using the untyped λ-calculus as follows. First, he introduced the operator $\star$ on λ-terms $t \star t' := \lambda_x.(t(t'x))$ .. Then, he went on defining

$$0 := \lambda_x.(\lambda_x.x), \quad 1 := \lambda_x.x, \quad 2 := \lambda_x.(x \star x), \ldots$$

so that $1f = f$, $2f = f \star f$, etc. as if a numeral n encodes the process of applying the $\star$ operator n times to its argument f.

1. What is $0f$? Is it expected? Comment.
2. What does the $\star$ operator encode as a standard operation on natural numbers?
3. What does $\lambda_x(x \star (nx))$ represent for a numeral n? (Hint: think of the basic ingredients you need to define a natural number object).
4. How can you encode exponentiation of natural numbers?
5. Bonus: Write the sum of two numerals as a λ-term.

So you might think that, since numerals behave like natural numbers, there is a built-in 'type' (precisely the one for natural numbers) that comes for free in an untyped λ-calculus... let's push this further yet before concluding.

1. Suppose that x has type A in the definition of $\star$, what would be the type of n and m in $n \star m$?
2. Is this type coherent with the product of two numerals? What about exponentiation of numerals?
3. State under which extra condition on the type A all numerals would have the same type. (So after all, numerals aren't really natural numbers...)