TD1

Khalil Ghorbal

1 Universal Property

Universal property literally means “for all such and such, there exists a unique foo-bar satisfying some stuff”.

Example 1 Assume all rings have a multiplicative identity called 1.

- Prove that the ring $\mathbb{Z}$ has the following property: for any ring R, there is a unique ring homomorphism $\mathbb{Z} \to R$.
- Let A be such that for any ring R, there is a unique ring homomorphism $A \to R$. Prove that $A \cong \mathbb{Z}$ (that is A is isomorphic to $\mathbb{Z}$).

Example 2 Given vector spaces U, V and W, a bilinear map $f : U \times V \to W$ is a map that is linear in each variable, that is

$$\begin{aligned} f(u_1 + \lambda u_2, v) &= f(u_1, v) + \lambda f(u_2, v) \\ f(u, v_1 + \lambda v_2) &= f(u, v_1) + \lambda f(u, v_2) \end{aligned}$$

for all $u, u_1, u_2 \in U$, and $v, v_1, v_2 \in V$ and scalars λ. (Think of the scalar or cross products as familiar examples.) We will say that the pair (T, b), where T is a vector space and $b : U \times V \to T$ is a bilinear map, is *universal* if it satisfies the following property

$$\begin{tikzcd} U \times V \arrow[r, "b"] \arrow[dr, "\forall\,\text{bilinear } f"'] & T \arrow[d, dotted, "\exists!\,\text{linear } \bar{f}"] \\ & \forall W \end{tikzcd}$$

In words: bilinear maps out of $U \times V$ are in one-to-one correspondence with linear maps out of T. Fix U and V and suppose (T, b) and (T', b') are both universal. Prove that there exists a unique isomorphism $j : T \to T'$ such that $j \circ b = b'$. (The unique object T is the tensor product of U and V denoted by $U \otimes V$.)

2 Categories

Exercise 1. Show that a map in a category can have only one inverse. That is if f is a map between A and B, there can be at most one map $g : B \to A$ such that $1_A = g \circ f$ and $1_B = f \circ g$.

Exercise 2. Write formally what’s missing to complete the formal definition of the product of two categories. (We have barely defined objects and maps during the course.)

3 Functors

Exercise 3. Show that functors preserve isomorphism. That is, if $F : \mathscr{A} \to \mathscr{B}$ is functor and $A \cong A'$ in $\mathscr{A}$, then $F(A) \cong F(A')$ in $\mathscr{B}$.

Exercise 4. Let $F : \mathscr{A} \times \mathscr{B} \to \mathscr{C}$ be a functor. Prove that for each $A \in \mathscr{A}$, there is a functor $F^A : \mathscr{B} \to \mathscr{C}$ defined on objects $B \in \mathscr{B}$ by $F^A(B) = F(A, B)$ and on maps g in $\mathscr{B}$ by $F^A(g) = F(1_A, g)$. Prove that for each $B \in \mathscr{B}$, there is a functor $F_B : \mathscr{A} \to \mathscr{C}$ defined similarly.

Exercise 5. Show that the families of functors $(F^A)_{A\in\mathscr{A}}$ and $(F_B)_{B\in\mathscr{B}}$ (w.r.t. the notations of the previous exercise) satisfy the following two conditions:

- if $A \in \mathscr{A}$ and $B \in \mathscr{B}$ then $F^A(B) = F_B(A)$;
- if $f : A \to A'$ in $\mathscr{A}$ and $g : B \to B'$ in $\mathscr{B}$ then $F^{A'}(g) \circ F_B(f) = F_{B'}(f) \circ F^A(g)$.

Exercise 6. Let $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ denote three categories and suppose that the families $(F^A)_{A\in\mathscr{A}}$ and $(F_B)_{B\in\mathscr{B}}$ satisfy the conditions of exercise 5. Prove that there is a unique functor $F : \mathscr{A} \times \mathscr{B} \to \mathscr{C}$ satisfying the equations in exercise 4.

4 Duality (bonus)

Let k denote a field. For any two vector spaces V and W over k, the set of linear maps $V \to W$ defines itself a vector space over k, denote $\mathbf{Hom}(V, W)$, where addition and scalar multiplication are defined pointwise. That is, for $f, g \in \mathbf{Hom}(V, W)$, $f + g$ is defined as the map $x \mapsto f(x) + g(x)$, for any $x \in V$.

Fix a vector space W. Any linear map $f : V \to V'$ induces a linear map $f^* : \mathbf{Hom}(V', W) \to \mathbf{Hom}(V, W)$.

- Define f^* using composition. (Notice that V and V' were inverted).
- Define a contravariant functor $\mathbf{Hom}(-, W)$ on Vect_k where the dash is a placeholder for the vector space. (That is a covariant functor $\mathrm{Vect}_k^{op} \to \mathrm{Vect}_k$.)
- When W is fixed to the special vector space k, what is the standard name given to $\mathbf{Hom}(-, k)$ in linear algebra?