Formulas as Types

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Cartesian closed categories

» Product

Let $\mathscr{A}$ be a category and $A, B \in \mathscr{A}$. A product of A and B consists of an object P and maps π, π' called projections

$$A \xleftarrow{\pi} P \xrightarrow{\pi'} B$$

such that for all triples (X, f, g) satisfying

$$A \xleftarrow{f} X \xrightarrow{g} B$$

there is a unique map $p : X \to P$ making the following diagram commute

$$\begin{tikzcd}
& X \arrow[dl, "f"'] \arrow[d, dashed, "p"] \arrow[dr, "g"] & \\
& P \arrow[dl, "\pi"] \arrow[dr, "\pi'"'] & \\
A & & B
\end{tikzcd}$$

» Product

Remarks

- P is often denoted by $A \times B$
- p is denoted by $\langle f, g \rangle$
- Products do not always exist!
- When a product exists, it induces a product-like operation on arrows $f \times g := \langle f \circ \pi, g \circ \pi' \rangle$ with

$$\begin{tikzcd}
& A \times B \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] \arrow[dd, dashed, "f\times g"] & \\
A \arrow[d, "f"'] & & B \arrow[d, "g"] \\
C & C \times D \arrow[l] \arrow[r] & D
\end{tikzcd}$$

- One can make sense of the product of zero elements. It is a terminal object! denoted 1 for convenience.

» Cartesian category

A category $\mathscr{A}$ is cartesian if it has finite products (including the product of zero elements 1). That is for every $A, B \in \mathscr{A}$, the product $A \times B$ exists.

» Cartesian closed category

Let $\mathscr{A}$ be a cartesian category. For every object $B \in \mathscr{A}$, we define a functor $- \times B : \mathscr{A} \to \mathscr{A}$ mapping object A to $A \times B$, and

$$A \xrightarrow{f} A' \quad \mapsto \quad A \times B \xrightarrow{\langle f \circ \pi, 1_B \circ \pi' \rangle} A' \times B$$

Cartesian closed category

A category $\mathscr{A}$ is cartesian closed if it is cartesian and for each $B \in \mathscr{A}$, the functor $- \times B : \mathscr{A} \to \mathscr{A}$ has a right adjoint.
We write the right adjoint as $(-)^B$, and, for $C \in \mathscr{A}$, call C^B an exponential.

So cartesian closed categories are those categories with products and exponentials.

» Important correspondences

In any cartesian closed category $\mathscr{A}$, and $A, B, C \in \mathscr{A}$, by definition of adjunctions

$$\mathscr{A}(A \times B, C) \cong \mathscr{A}(A, C^B)$$

that is arrows $A \times B \to C$ are in one-to-one correspondence with arrows $A \to C^B$. We called such operation a transposition and denoted it by a bar in both directions. We can also prove $(A = 1)$

$$\mathscr{A}(B, C) \cong \mathscr{A}(1, C^B)$$

so that to each $B \xrightarrow{g} C$ corresponds $1 \xrightarrow{\overline{g \circ \pi'}} C^B$ and to each $1 \xrightarrow{f} C^B$ corresponds $B \xrightarrow{\varepsilon_C \circ f \times 1_B} C$.

» Higher-order arithmetic

Looks familiar?

In any cartesian category $\mathscr{A}$, and $A, B, C \in \mathscr{A}$

$$A \times 1 \cong A, \quad A \times B \cong B \times A, \quad (A \times B) \times C \cong A \times (B \times C)$$

In any cartesian closed category $\mathscr{A}$, and $A, B, C \in \mathscr{A}$

$$A^1 \cong A, \quad 1^A \cong 1, \quad (A \times B)^C \cong A^C \times B^C, \quad A^{B \times C} \cong (A^C)^B$$

» A category of cartesian closed categories

A cartesian functor $F : \mathscr{A} \to \mathscr{B}$ is a functor that preserves the cartesian closed structure

* $F(A \times B) = F(A) \times F(B)$
* $F(\pi_{A,B}) = \pi_{F(A),F(B)}$
* $F(A^B) = F(A)^{F(B)}$
* $F(\langle f, g \rangle) = \langle F(f), F(g) \rangle$
* ...

This defines a category Cart of cartesian closed categories.

» Monad

A monad on a category $\mathscr{A}$ is a triple (T, η, μ) where $T : \mathscr{A} \to \mathscr{A}$

is a functor equipped with a unit
$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "1_{\mathscr{A}}"] \arrow[r, bend right, "T"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathscr{A} \end{tikzcd}$$

and a multiplication
$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "T \circ T"] \arrow[r, bend right, "T"'] \arrow[r, phantom, "\Downarrow\mu"] & \mathscr{A} \end{tikzcd}$$

satisfying the associativity and unit laws. That is such that the following diagrams commute:

$$\begin{tikzcd} T \circ T \circ T \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T \circ T \arrow[d, "\mu"] \\ T \circ T \arrow[r, "\mu"'] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "T\eta"] \arrow[dr, "1_T"'] & T \circ T \arrow[d, "\mu"] \\ & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "\eta T"] \arrow[dr, "1_T"'] & T \circ T \arrow[d, "\mu"] \\ & T \end{tikzcd}$$

Every adjunction defines a monad!

» Kleisli category

The Kleisli category $\mathscr{A}_T$ of a monad (T, η, μ) on $\mathscr{A}$ is a category with

- the same objects as $\mathscr{A}$
- with morphisms $A \twoheadrightarrow B$ whenever $A \to TB$ is a morphism in $\mathscr{A}$

The identity arrow $1_A : A \twoheadrightarrow A$ is defined as $\eta_A : A \to TA$.
Two morphisms $f : A \twoheadrightarrow B$ and $g : B \twoheadrightarrow C$ are composed as $\mu_C \circ Tg \circ f$:

$$
\begin{tikzcd}
 & & TTC \arrow[d, "\mu_C"] \\
 & TB \arrow[ur, "Tg"'] & TC \\
A \arrow[ur, "f"'] & B \arrow[ur, "g"] &
\end{tikzcd}
$$

» Natural numbers objects

A natural numbers object (or system) in a cartesian closed category $\mathscr{A}$ is an object N and two maps

$$1 \xrightarrow{0} N \xrightarrow{s} N$$

satisfying the following universal property: for any diagram

$$1 \xrightarrow{a} A \xrightarrow{f} A$$

there is a unique arrow $N \xrightarrow{h} A$ such that $h \circ 0 = a$ and $h \circ s = f \circ h$. That is such the following diagram commutes

$$\begin{tikzcd} 1 \arrow[r, "0"] \arrow[dr, "a"'] & N \arrow[r, "s"] \arrow[d, dashed, "h"] & N \arrow[d, dashed, "h"] \\ & A \arrow[r, "f"] & A \end{tikzcd}$$

Drop uniqueness of h to get weak natural numbers object.

Deductive systems

» Deductive system

A deductive system is a category without the associativity and identity laws axioms.

* objects are called formulas
* arrows are called proofs

» Conjunction calculus

A conjunction calculus is a deductive system with

- $\top$ a formula $\top$ (called "true") such that there is an arrow $A\bullet : A \to \top$ for each object (a terminal-like object … but we don't have a category)
- $\wedge$ a binary operation '$\wedge$' between formulas (called "conjunction") together with two arrows $A \wedge B \xrightarrow{\pi_{A,B}} A$ and $A \wedge B \xrightarrow{\pi'_{A,B}} B$ inducing a *pairing* of arrows with the same domain often presented as an inference rule

$$\frac{C \xrightarrow{f} A \quad C \xrightarrow{g} B}{C \xrightarrow{\langle f,g \rangle} A \wedge B}$$

(a product-like construction)

» Proof calculus

proving means *constructing* new proofs (arrows) from a formula (assumption) to another formula (result)

For instance, in conjunction calculus, $\wedge$ is commutative and associative (the labels on arrows are the proofs)

- $A \wedge B \xrightarrow{\langle \pi'_{A,B}, \pi_{A,B} \rangle} B \wedge A$
- $(A \wedge B) \wedge C \xrightarrow{\alpha_{A,B,C}} A \wedge (B \wedge C)$ where $\alpha_{A,B,C} = \langle \pi_{A,B} \circ \pi_{A \wedge B,C}, \langle \pi'_{A,B} \circ \pi_{A \wedge B,C}, \pi'_{A \wedge B,C} \rangle \rangle$

» Proof calculus

Inference rules define a calculus over proofs: for instance conjunction of formulas ($\wedge$) induces an operation over arrows (pairing).

Other operations on proofs can be defined out of known ones (derived rules).

For instance

$$\frac{A \xrightarrow{f} B \quad C \xrightarrow{g} D}{A \wedge C \xrightarrow{\langle f \circ \pi_{A,C}, g \circ \pi'_{A,C} \rangle} B \wedge D}$$

defines a “conjunction” on proofs:

$$f \wedge g := \langle f \circ \pi_{A,C}, g \circ \pi'_{A,C} \rangle$$

» Positive intuitionistic propositional calculus

A positive intuitionistic propositional calculus is a conjunction calculus with an additional binary operation between formulas

- a binary operation '$\Leftarrow$' between formulas (called "if") together with an arrow $(A \Leftarrow B) \wedge B \xrightarrow{\varepsilon_{A,B}} A$ inducing the following *transposition* on arrows:

$$\frac{C \wedge B \xrightarrow{h} A}{C \xrightarrow{\overline{h}} A \Leftarrow B}$$

» Associated proof calculus

One derives two operations on proofs

$$\frac{A \xrightarrow{f} B}{\top \xrightarrow{\overline{f \circ \pi'_{\top,A}}} B \Leftarrow A}$$

$$\frac{\top \xrightarrow{g} B \Leftarrow A}{A \xrightarrow{\varepsilon_{B,A} \circ \langle g \circ A\bullet, 1_A \rangle} B}$$

We denote

$$f_* := \overline{f \circ \pi'_{\top,A}} \qquad g^* := \varepsilon_{B,A} \circ \langle g \circ A\bullet, 1_A \rangle$$

f_* is called the name of f.

» Deduction theorem

Ring like construction

Given a positive intuitionistic calculus $\mathscr{D}$, assuming the proof $\top \xrightarrow{x} A$, one gets a new positive intuitionistic calculus $\mathscr{D}(x)$ with the same formulas as $\mathscr{D}$ and where the proofs, called polynomials, are freely generated using the induced operators on proofs (inference and derived rules), like $\langle -, - \rangle$, $\wedge$, $(-)^*$ and $(-)_*$

Deduction theorem on proofs

With every proof $B \xrightarrow{\varphi(x)} C$ in $\mathscr{D}(x)$ from the *assumption* $\top \xrightarrow{x} A$, there exists an associated proof $A \wedge B \xrightarrow{f} C$ in $\mathscr{D}$ not depending on x.

» Other deduction systems

One can go further and define

- intuitionistic propositional calculus (adding falsehood and disjunction)
- classical propositional calculus (adding the excluded middle)

» Deduction systems as categories

We can fully recover a category structure from a deduction system by adding back the missing axioms as equivalence relations between proofs.
More precisely, the equality between proofs is decided modulo the following identities

* $f \circ 1_A = f$, for any object A and arrow f with domain A
* $1_A \circ f = f$, for any object A and arrow f with codomain A
* $(f \circ g) \circ h = f \circ (g \circ h)$, for any composable arrows f, g, h

» Conjunction calculus as cartesian category

Conjunction calculus can be regarded as a cartesian category by restricting further the equality between proofs modulo the following identities:

* $f = A\bullet$, for any $A \xrightarrow{f} \top$ (now $\top$ becomes a terminal object)
* for all $C \xrightarrow{f} A$, $C \xrightarrow{g} B$, $C \xrightarrow{h} A \wedge B$
  * $\pi_{A,B} \circ \langle f, g \rangle = f$
  * $\pi'_{A,B} \circ \langle f, g \rangle = g$
  * $\langle \pi_{A,B} \circ h, \pi'_{A,B} \circ h \rangle = h$
* This turns the conjunction into a product

» Positive intuitionistic calculus as cartesian closed category

We restrict further the equalities between proofs modulo the following identities:

for all $C \wedge B \xrightarrow{h} A$ and $C \xrightarrow{k} A \Leftarrow B$

* $\varepsilon_{A,B} \langle \overline{h} \circ \pi_{C,B}, {\pi'}_{C,B} \rangle = h$
* $\overline{\varepsilon_{A,B} \circ \langle k \circ \pi_{C,B}, {\pi'}_{C,B} \rangle} = k$

These identities make the “if” binary operation ‘$\Leftarrow$’ into an exponential, so $B \Leftarrow A$ defines B^A which satisfies all the properties of a (right) adjunction for the product functor.

» Polynomial category

Let $\mathscr{A}$ denote the cartesian closed category obtained from a positive intuitionistic calculus $\mathscr{D}$.

The polynomial category $\mathscr{A}[x]$ is defined as the cartesian closed category obtained from the associated positive intuitionistic calculus $\mathscr{D}(x)$ assuming $\top \xrightarrow{x} A$.

Remark: The category $\mathscr{A}[x]$ is isomorphic to a Kleisli category.

» A deduction theorem over categories

For any polynomial $\varphi(x) : B \to C$ in $\mathscr{A}[x]$ there is a unique arrow $f : A \times B \to C$ in $\mathscr{A}$ such that $f \circ \langle x \circ B\bullet, 1_B \rangle = \varphi(x)$. (The equality here is between equivalence classes by construction of $\mathscr{A}$ and $\mathscr{A}[x]$.)

This says that polynomials have very special canonical form, very much like $a_0 + a_1 X + a_2 X^2 + \cdots$ is the canonical form of univariate polynomials over a ring of coefficients.

(NB: nothing says that the arrow f is simple!)

» Functional completeness

For any polynomial $\varphi(x) : \top \to B$ in $\mathscr{A}[x]$, where $x : \top \to A$ is an assumption in $\mathscr{A}$, there is a unique arrow $f : \top \to B^A$ in $\mathscr{A}$ such that $\varepsilon_{B,A} \circ \langle f, x\rangle = \varphi(x)$ (again the equality is over equivalence classes). We denote f by $\lambda_{x:A}\varphi(x)$.

Yes, this will be the λ abstraction in the typed λ-calculus.

λ-calculus

» Untyped $\boldsymbol{\lambda}$-calculus

Combinatory logic

The pure λ-calculus is a formal language. Its words, called λ-terms are defined inductively

$$t ::= x \mid t \wr t' \mid \lambda_x.t$$

where the (total) binary operator $\wr$, called application and the binder λ (over variables), called λ-abstraction satisfy the following axioms:

(β) $(\lambda_x.\varphi(x)) \wr a = \varphi(a)$, whenever no free occurrence in a becomes bound in $\varphi(a)$; we say x is substituted by a, or a is substitutable for x.

(η) $\lambda_x.(f \wr x) = f$, whenever f is independent from x (i.e. if x occurs in f it must be bound).

A term is closed if it contains no free variables.

» α-renaming

Equivalence relation

Terms are considered equal up to renaming their bound variables.
This defines a congruence relation (equality over equivalence classes).
For instance $\lambda_x.y \equiv \lambda_z.y \not\equiv \lambda_y.y$

» Typed λ-calculus

A typed λ-calculus is a formal language consisting of

* a class of types
* a class of terms for each type

The class of types

* has some basic types (like $\top$, or N for natural numbers)
* is closed under products and exponentials: for any types A and B, $A \times B$ and B^A are also types.

The class of terms is freely generated

* from variables of certain types
* term forming operations: pairing $\langle -, - \rangle$, projections π, π', evaluation $\varepsilon_{A,B}$, and λ-abstraction

» Translations

A translation is a morphism over λ-calculi $\phi : \mathscr{L} \to \mathscr{L}'$:

* $\phi(A)$ is a type of $\mathscr{L}'$ for any type A of $\mathscr{L}$
* $\phi(1) = 1$, $\phi(A \times B) = \phi(A) \times \phi(B)$ etc.
* for every arrow $a : 1 \to A$ in $\mathscr{L}$, $\phi(a) : 1 \to \phi(A)$ in $\mathscr{L}'$
* if a is closed, then $\phi(a)$ is closed

This defines a category λ-Calc of λ-calculi.

» Internal language of cartesian closed categories

Let $\mathscr{A}$ denote a cartesian closed categories.
The internal language $L(\mathscr{A})$ of $\mathscr{A}$ is defined by

- types are the formulas of $\mathscr{A}$
- terms of type A are polynomial expressions $\varphi(x) : 1 \to A$ in the polynomial category $\mathscr{A}[x]$ where $x : 1 \to B$ (typed variables).
  Notice that the domain for terms is the terminal object 1. Thus any arrow in the polynomial category is not a term, but its name is (cf. slide 14).
- (We need a natural numbers object to have multiple variables.)

Terms are "ordinary elements" of types.

» Curry-Howard-Lambek isomorphism

The internal language construction defines a functor

$L : \mathbf{Cart} \to \lambda\text{-}\mathbf{Calc}$

We can also generate a cartesian closed category from a λ-calculus. This defines a functor $C : \lambda\text{-}\mathbf{Calc} \to \mathbf{Cart}$

Curry-Howard-Lambek isomorphism

$\lambda\text{-}\mathbf{Calc} \cong \mathbf{Cart}$

(The equivalence still stand if one adds a (weak) natural number object in both the language and the cartesian closed category.)

» This is only the big bang

You have already seen *a* lot up to this point.

But that's just the beginning!

* The construction of $\mathscr{A}[x]$ using Kleisli categories
* Monads and algebraic theories
* reduction and bounded (strongly normalizing) λ-terms (coherence problem)
* C-monoidal categories and untyped λ-calculus
* ...

To be continued ...

» References

* Joachim Lambek, Philip J. Scott. Introduction to Higher-Order Categorical Logic. 1986
* Steve Awodey, Category Theory. 2009