Category Theory

Crash Course

by Khalil Ghorbal (Inria, France)
on December 7, 2023

Categories

» Category

A category $\mathscr{A}$ consists of

* a collection of objects, $\mathrm{ob}(\mathscr{A})$;
* for any two objects, $A, B \in \mathrm{ob}(\mathscr{A})$, a collection of maps or arrows $\mathscr{A}(A, B)$ from A to B;
* for any three objects, A, B, C, a binary operator on maps called composition $\circ : \mathscr{A}(B, C) \times \mathscr{A}(A, B) \to \mathscr{A}(A, C)$;
* for each object A an identity map 1_A in $\mathscr{A}(A, A)$,

satisfying the following axioms

* Associativity: for any $f \in \mathscr{A}(A, B)$, $g \in \mathscr{A}(B, C)$, $h \in \mathscr{A}(C, D)$, $(h \circ g) \circ f = h \circ (g \circ f)$,
* Identity Laws: for any $f \in \mathscr{A}(A, B)$, $f \circ 1_A = f = 1_B \circ f$.

» Lingo

* $f : A \longrightarrow B$ or $A \xrightarrow{f} B$ mean $f \in \mathscr{A}(A, B)$
* A is called the domain of f
* B is called the codomain of f
* A map $f \in \mathscr{A}(A, B)$ is an isomorphism if there exists a map $g \in \mathscr{A}(B, A)$ such that $f \circ g = 1_B$ and $f \circ g = 1_A$
* if $f \in \mathscr{A}(A, B)$ is an isomorphism, we say that A and B are isomorphic and write $A \cong B$
* a diagram commutes means that any two maps with the same domain and codomain are equal.

The following diagram commutes means $h = g \circ f$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

» Examples of categories

- There is a category with no objects (and no maps)
- $\bullet$ (one object and one map $1_\bullet$)
- $\bullet \longrightarrow \bullet$ (two objects and one map between them)

- **Set**: sets and functions
- **Grp**: groups and group homomorphisms
- **Ring**: rings and ring homomorphisms
- **Vect**$_k$: vector spaces over the field k and linear maps
- **Top**: topological spaces and continuous maps

» Opposite category

Every category $\mathscr{A}$ has an opposite category $\mathscr{A}^{\mathrm{op}}$ obtained from $\mathscr{A}$ by reversing its arrows:

* $\mathscr{A}$ and $\mathscr{A}^{\mathrm{op}}$ have the same objects and identity elements
* for any two objects A, B, $\mathscr{A}^{\mathrm{op}}(B, A) = \mathscr{A}(A, B)$.

Opposite group

If $\mathscr{G}$ denotes the group G seen as a category (one object and isomorphic maps to and from that object), then $\mathscr{G}^{\mathrm{op}}$ is the opposite group of G.

» Product category

Given categories $\mathscr{A}$ and $\mathscr{B}$, a product category $\mathscr{A} \times \mathscr{B}$ is defined by

* $\mathrm{ob}(\mathscr{A} \times \mathscr{B}) = \mathrm{ob}(\mathscr{A}) \times \mathrm{ob}(\mathscr{B})$,
* $(\mathscr{A} \times \mathscr{B})((A, B), (A', B')) = \mathscr{A}(A, A') \times \mathscr{B}(B, B')$.

The notation $\mathrm{ob}(\mathscr{A}) \times \mathrm{ob}(\mathscr{B})$ is simply a shorthand notation to say that objects of the product category is a pair of objects. Likewise for maps.

Functors

» Functor

Let $\mathscr{A}$ and $\mathscr{B}$ be categories. A covariant functor $F : \mathscr{A} \to \mathscr{B}$ consists of

* a mapping from $\mathrm{ob}(\mathscr{A})$ to $\mathrm{ob}(\mathscr{B})$ that takes any object A of $\mathscr{A}$ to an object $F(A)$ (or simply FA) of $\mathscr{B}$;
* for any two objects, A, A' of $\mathrm{ob}(\mathscr{A})$, a mapping from $\mathscr{A}(A, A')$ to $\mathscr{B}(F(A), F(A'))$ that takes f to $F(f)$ (or Ff),

satisfying the following axioms

* $F(f' \circ f) = F(f') \circ F(f)$ for any $f \in \mathscr{A}(A, A')$ and $f' \in \mathscr{A}(A', A'')$,
* $F(1_A) = 1_{FA}$ for any $A \in \mathscr{A}$.

A contravariant functor from $\mathscr{A}$ to $\mathscr{B}$ is a functor $\mathscr{A}^{\mathrm{op}} \to \mathscr{B}$.

» Schema

Let $p_1, \ldots, p_m$ denote some polynomials in $A[X_1, \ldots, X_n]$ for some ring A. Let $F(A)$ denote the set of common roots in A^n to all polynomials p_i. Then F can be regarded as (covariant) functor from Ring to Set. If f is a ring homomorphism from A to B, and $(r_1, \ldots, r_n)$ is an element of $F(A)$, then $(f(r_1), \ldots, f(r_n))$ is an element of $F(B)$. $F(f)$ is a well defined function from A^n to B^n.

F is called the schema (associated with the considered system of polynomials).

» Duality

Let X be a topological space and let $C(X)$ denote the set of real-valued functions defined on X. Let f be a continuous function from X to Y, then we define $C(f)$ as a function from $C(Y)$ to $C(X)$ (notice the inversion) as the composition

$$X \xrightarrow{f} Y \xrightarrow{q} \mathbb{R} \quad .$$

Observe that $q \in C(Y)$ and that the composition defines in turn an element of $C(X)$.

A presheaf on $\mathscr{A}$ is a contravariant functor from $\mathscr{A}$ to **Set** (like C above where $\mathscr{A}$ is **Top**).

» Faithful / Full functor

A functor $F : \mathscr{A} \to \mathscr{B}$ is faithful (resp. full) if for each $A, A' \in \mathscr{A}$, the function

$$\begin{aligned} \mathscr{A}(A, A') &\to \mathscr{B}(F(A), F(A')) \\ f &\mapsto F(f) \end{aligned}$$

is injective (resp. surjective).

Careful. Injectivity and surjectivity are not considered on any arrows of the category $\mathscr{A}$ but with respect to a pair of elements in $\mathscr{A}$.

» Subcategory

Let $\mathscr{A}$ be a category. A subcategory $\mathscr{S}$ of $\mathscr{A}$ consists of a subclass of $\mathrm{ob}(\mathscr{A})$ together with, for each pair $S, S' \in \mathrm{ob}(\mathscr{S})$, a subclass of $\mathscr{A}(S, S')$ such as $\mathscr{S}$ is closed under composition and identities. It is a full subcategory if $\mathscr{S}(S, S') = \mathscr{A}(S, S')$ for all $S, S' \in \mathrm{ob}(\mathscr{S})$.

Careful. The image of a functor $F : \mathscr{A} \to \mathscr{B}$ needs not be a subcategory of $\mathscr{B}$.

Natural Transformations

» Natural transformation

Let $\mathscr{A}, \mathscr{B}$ be categories and let $\mathscr{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathscr{B}$ be two functors.

A natural transformation $\alpha : F \to G$ is a family $\left(F(A) \xrightarrow{\alpha_A} G(A)\right)_{A \in \mathscr{A}}$ of maps in $\mathscr{B}$ such that for every map $A \xrightarrow{f} A'$ in $\mathscr{A}$, the square

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] & F(A') \arrow[d, "\alpha_{A'}"] \\ G(A) \arrow[r, "G(f)"'] & G(A') \end{tikzcd}$$

commutes. The maps α_A are called the components of α.

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathscr{B} \end{tikzcd}$$

» Functor category

- $\begin{tikzcd} \mathscr{A} \arrow[r, bend left=60, "F"] \arrow[r, "G" description] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & \mathscr{B} \end{tikzcd}$ can be composed $\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Downarrow\beta\circ\alpha"] & \mathscr{B} \end{tikzcd}$.
- $\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Downarrow 1_F"] & \mathscr{B} \end{tikzcd}$, defined by $(1_F)_A = 1_{F(A)}$ for each $A \in \mathscr{A}$.

For any two categories $\mathscr{A}$ and $\mathscr{B}$, the functor category, denoted $[\mathscr{A}, \mathscr{B}]$ or $\mathscr{B}^{\mathscr{A}}$, is a category whose objects are the functors $\mathscr{A} \to \mathscr{B}$ and whose maps are the natural transformations between them.

» Natural isomorphism

Let $\mathscr{A}$ and $\mathscr{B}$ be two categories. A natural isomorphism between functors $F, G : \mathscr{A} \to \mathscr{B}$ is an isomorphism in $[\mathscr{A}, \mathscr{B}]$. We say that F and G are naturally isomorphic and write $F \cong G$.

Lemma – equivalent definition

Let $\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \mathscr{B} \arrow[from=U, to=D, Rightarrow, "\alpha"] \end{tikzcd}$ be a natural transformation.

Then α is a natural isomorphism if and only if $\alpha_A : F(A) \to G(A)$ is an isomorphism for each $A \in \mathscr{A}$.

When $F \cong G$, we also say that $F(A) \cong G(A)$ naturally in A.

» Equivalent categories

An equivalence between categories $\mathscr{A}$ and $\mathscr{B}$ consists of a pair of functors $\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}$ together with natural isomorphisms η and ε

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "1_{\mathscr{A}}"] \arrow[r, bend right, "G\circ F"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathscr{A} \end{tikzcd} \quad , \quad \begin{tikzcd} \mathscr{B} \arrow[r, bend left, "F\circ G"] \arrow[r, bend right, "1_{\mathscr{B}}"'] \arrow[r, phantom, "\Downarrow\varepsilon"] & \mathscr{B} \end{tikzcd} \quad .$$

We say that $\mathscr{A}$ and $\mathscr{B}$ are equivalent, denoted $\mathscr{A} \simeq \mathscr{B}$, and that the functors F and G are equivalences.

Careful. Equivalence of categories ($\mathscr{A} \simeq \mathscr{B}$) is *weaker* than isomorphisms of categories ($\mathscr{A} \cong \mathscr{B}$) — as elements of CAT.

Remark. An equivalence of the form $\mathscr{A}^{\mathrm{op}} \simeq \mathscr{B}$ is sometimes called a duality between $\mathscr{A}$ and $\mathscr{B}$.

» Sameness

- Equality for elements of a set
- Isomorphism for objects of a category
- Natural isomorphism for functors
- Equivalence for categories

» Ajoints

Let $\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}$ be categories and functors. We say that F is left adjoint to G, and G is right adjoint to F, and write $F \dashv G$ if

1. For every $A \in \mathscr{A}$ and $B \in \mathscr{B}$, there is a one-to-one correspondence, called an adjunction, between $\mathscr{A}(A, G(B))$ and $\mathscr{B}(F(A), B)$.
   Notation: For each $f : A \to G(B)$ (resp. $g : F(A) \to B$), its corresponding map (w.r.t. to an adjunction) is denoted by $\overline{f}$ (resp. $\overline{g}$) and called *the* transpose of f (resp. g). So one has $\overline{\overline{f}} = f$ for all $f \in \mathscr{A}(A, G(B))$ and $\overline{\overline{g}} = g$ for all $g \in \mathscr{B}(F(A), B)$.

2. The following naturality axioms are satisfied
   * $\overline{q \circ g} = G(q) \circ \overline{g}$ for all $g \in \mathscr{B}(F(A), B)$ and arrow q in $\mathscr{B}$ with domain B
   * $\overline{f \circ p} = \overline{f} \circ F(p)$ for all $f \in \mathscr{A}(A, G(B))$ and arrow p in $\mathscr{A}$ with codomain A

» Example: "Currification"

A map $A \times B \to C$ can be seen as a map $A \to C^B$ (or $A \to (B \to C)$)

Fix an object B in Set the category of sets. Consider the two following functors (cf. next slides for the formal definitions of product and exponentiation of sets.)

$$
\begin{aligned}
- \times B : \mathsf{Set} &\to \mathsf{Set} \\
A &\mapsto A \times B
\end{aligned}
\qquad\qquad
\begin{aligned}
(-)^B : \mathsf{Set} &\to \mathsf{Set} \\
C &\mapsto C^B
\end{aligned}
$$

Then $(- \times B)$ is left adjoint to $(-)^B$:

$$
\begin{tikzcd}
\mathsf{Set} \arrow[r, shift left, "- \times B"] \arrow[r, phantom, "\perp"] & \mathsf{Set} \arrow[l, shift left, "(-)^B"]
\end{tikzcd}
$$

» Product

Let $\mathscr{A}$ be a category and $A, B \in \mathscr{A}$. A product of A and B consists of an object $A \times B$ and maps π, π' called projections

$$A \xleftarrow{\pi} A \times B \xrightarrow{\pi'} B$$

such that for all P, f, g satisfying

$$A \xleftarrow{f} P \xrightarrow{g} B$$

there is a unique map $p : P \to A \times B$, denoted (f, g), making the following diagram commute

$$\begin{tikzcd} & P \arrow[dl, "f"'] \arrow[d, dashed, "p"] \arrow[dr, "g"] & \\ A & A \times B \arrow[l, "\pi"] \arrow[r, "\pi'"'] & B \end{tikzcd}$$

» Exponential ‘Function Set’

Let $\mathscr{A}$ be a category and $A, B \in \mathscr{A}$. An Exponential from A to B consists of an object F (often denoted B^A) together with a map (called sometimes evaluation)

$$F \times A \xrightarrow{\varepsilon} B$$

such that for all objects X and maps satisfying

$$X \times A \xrightarrow{q} B$$

there is a unique map $\bar{q} : X \to F$ making the following diagram commute ($\varepsilon \circ (\bar{q} \times 1_B) = q$)

$$\begin{tikzcd} X \times A \arrow[d, dashed, "\bar{q}\times 1_B"'] \arrow[dr, "q"] & \\ F \times A \arrow[r, "\varepsilon"] & B \end{tikzcd}$$

» Unit/Counit

Let $\mathscr{A} \underset{G}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathscr{B}$. Let $A \in \mathscr{A}$ and $B \in \mathscr{B}$. Fix an adjunction between $\mathscr{A}(A, G(B))$ and $\mathscr{B}(F(A), B)$. For every $A \in \mathscr{A}$, the transpose of $1_{F(A)}$ defines a map $\eta_A : A \to GF(A)$ and, dually, for every $B \in \mathscr{B}$, the transpose of $1_{G(B)}$ defines a map $\varepsilon_B : FG(B) \to B$.
These maps define natural transformations

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "1_{\mathscr{A}}"] \arrow[r, bend right, "G\circ F"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathscr{A} \end{tikzcd} \quad , \quad \begin{tikzcd} \mathscr{B} \arrow[r, bend left, "F\circ G"] \arrow[r, bend right, "1_{\mathscr{B}}"'] \arrow[r, phantom, "\Downarrow\varepsilon"] & \mathscr{B} \end{tikzcd} ,$$

called unit and counit of the adjunction respectively.

» Horizontal composition

Special case

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "1_{\mathscr{A}}"] \arrow[r, bend right, "G\circ F"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathscr{A} \arrow[r, "F"] & \mathscr{B} \end{tikzcd} \quad \text{defines} \quad \begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "F\circ G\circ F"'] \arrow[r, phantom, "\Downarrow F\eta"] & \mathscr{B} \end{tikzcd}$$

$$\begin{tikzcd} \mathscr{A} \arrow[r, "F"] & \mathscr{B} \arrow[r, bend left, "F\circ G"] \arrow[r, bend right, "1_{\mathscr{B}}"'] \arrow[r, phantom, "\Downarrow\varepsilon"] & \mathscr{B} \end{tikzcd} \quad \text{defines} \quad \begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F\circ G\circ F"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Downarrow \varepsilon F"] & \mathscr{B} \end{tikzcd}$$

» Triangle identities

Lemma

Given an adjunction $F \dashv G$ with unit η and counit ε, the following diagrams, called triangle identities commute

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "1_F"'] & FGF \arrow[d, "\varepsilon F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\eta G"] \arrow[dr, "1_G"'] & GFG \arrow[d, "G\varepsilon"] \\ & G \end{tikzcd}$$

» Transposes via unit/counit

Lemma

Let $\mathscr{A} \underset{G}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathscr{B}$ be an adjunction with unit η and counit ε. Then

$$\overline{g} = G(g) \circ \eta_A \qquad \text{and} \qquad \overline{f} = \varepsilon_B \circ F(f),$$

for any $g : F(A) \to B$ and $f : A \to G(B)$.

An adjunction is fully defined by its unit and counit.

Adjoints via unit/counit

Theorem

Let $\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}$ be categories and functors.
Then $F \dashv G$ if and only if there exist natural transformations $\eta : 1_{\mathscr{A}} \to GF$ and $\varepsilon : FG \to 1_{\mathscr{B}}$ satisfying the triangle identities.

This is an equivalent definition for adjoints.

» References

* Tom Leinster. Basic Category Theory. *arXiv version*. 2016